Asymptotic of generalized Bergman kernel on a non compact Kähler manifold

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ABSTRACT. Let E be a hermitian holomorphic vector bundle with bounded geometry and L a positive line bundle on M, a Kähler manifold with bounded geometry. We give an asymptotic expansion of the kernel over the diagonal of the orthogonal projection from the space of sections in $L^2(M, E\otimes L^k)$ on the space of holomorphic sections $H^0\left(M, E\otimes L^k\right)$ when $k\to\infty$. This gives a generalization of the work of Z. Lu.

1. Introduction

Let M be a smooth Kähler manifold of complex dimension n, equipped with a complete Kähler metric $g_{i\bar{j}}$ of bounded geometry of order $5+0$, and E be a holomorphic vector bundle on M of rank r, equipped with a hermitian metric of bounded geometry of order $3+0$ (the notions of bounded geometry will be explicitly given in section 2). Let $\omega=\omega_g=\frac{\sqrt{-1}}{2}\sum g_{i\bar{j}}dz_i\wedge d\overline{z_j}$ be the closed definite positive $(1,1)$-form associated to the metric $g_{i\bar{j}}$. We fix a polarization L on M, which means a line bundle on M and h_L a smooth hermitian metric on L such that its curvature $\Theta\left(L\right)$ is positive (i.e. the smaller eigenvalue of $\Theta\left(L\right)$ respectively to ω is positive on M). First of all, for the sake of clearty we assume that the following assumption holds :

$$\omega_g=-\frac{i}{2\pi}\partial\overline{\partial}\log\left(h_L\right). \tag{1.1}$$

With this setting, for all $k>0$, we get a natural metric $h_k=\langle .,.\rangle_{h_k}:=h_E\cdot h_L^k$ on the vector bundle $E\otimes L^k$ and also an endomorphism of vector bundle associated to the projection of the space of sections in $L^2\left(M, E\otimes L^k\right)$ on the space of holomorphic sections $H^0\left(M, E\otimes L^k\right)$,

$$\widehat{\mathsf{B}}:=\widehat{\mathsf{B}}_k=\sum_i S_i\langle .,S_i\rangle_{h_k}\in End(E) \tag{1.2}$$

where S_i is any hilbertian basis of $H^0\left(M, E\otimes L^k\right)$ for the inner product $\int_M h_k(.,.)dV$. The operator $\widehat{\mathsf{B}}_k$ will be denoted as the *generalized Bergman kernel* and we will compute explicitly its asymptotic expansion in the variable k up to order 2.

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For this, we build a family of peak sections, using Tian's method and the $\overline{\partial}$-operator resolution with values in a vector bundle, introduced by J-P Demailly for a complete Kähler manifold. Eventually, we complete our free family as a L^2 hilbertian basis in order to express in local coordinates the Bergman kernel, and we give some generalizations of our method.

Our goal is essentially to generalize the work of Z. Lu in [Lu, Theorem 1.1] i.e. the particular case of $E = L$, on a compact manifold by :

Theorem 1.1. *Let (M, ω) be a complete Kähler manifold with bounded geometry of order $5+0$ and (E, h_E) a hermitian holomorphic vector bundle with bounded geometry of order $3+0$. Let (L, h_L) a polarization on M satisfying equation (1.1) and such that $Ric(\omega_g) + C\omega_g > 0$ for a certain constant $C > 0$. Then we have the following asymptotic expansion for the generalized Bergman kernel on $E \otimes L^k$ when $k \to \infty$:*

$$\left\| \widehat{\mathsf{B}}_k - k^n Id_{r\times r} - \left(\frac{1}{2}Scal(g)Id_{r\times r} + \sqrt{-1}\Lambda\Theta\left(E\right)\right)k^{n-1}\right\|_{C^0} \leq C_0 k^{n-2} \tag{1.3}$$

where $Scal\left(g\right)$ is the scalar curvature of the metric associated to ω, and $\Theta\left(E\right)$ denotes the Chern curvature of the vector bundle E. This estimate is uniform on M when h_L (resp h_E) and its derivatives of order ≤ 5 (resp ≤ 3) belong to a compact set for the C^0 topology.

At that point, we want to emphasize the simplicity of the term of weight k^{n-1}. We have only established this expansion in the C^0 topology, but Theorem 1.1 should hold for all C^m topology $\forall m > 0$, assuming that the boundedness geometry is of higher order.

This work is part of the author's Ph.D thesis [Ke].

2. Background material

We recall in that section some standard notions and conventions of Kähler geometry. Some good references on this topic are [De2],[Ko],[T-Y] for more precise details.

Definition 2.1. A quasi-coordinate map is an holomorphic map ρ from an open ball B in $\mathbb{C}^n$ into a complex manifold M of dimension n of maximal rank everywhere. (B, ρ) is called a local quasi-coordinate for the manifold.

Definition 2.2. A Kähler metric on a complex manifold M has bounded geometry of order $k+\alpha$ (or is said to be regular homogeneous or quasi-finite) with $k \in \mathbb{N}$ and $0 \leq \alpha < 1$ if there exists a system (B_l, ρ_l) of local quasi-coordinates of M such that

1. Every point $z \in M$ is the image by ρ_l of the center of some ball B_l,
2. There are positive constants δ_1, δ_2, independent of l, such that the radius r_l of B_l satisfies $\delta_1 < r_l < \delta_2$,
3. There are positive constants C, C_k such that

$$\begin{array}{ll} 0 < \frac{1}{C}\delta_{ij} \leq g_{i\overline{j}}\left(l\right) \leq C\delta_{ij} & \\ \left|\frac{\partial^{|p|+|q|}}{\partial z^p \partial \overline{z}^q} g_{i\overline{j}}\left(l\right)\right|_{C^\alpha(B_l)} \leq C_{|p|+|q|} & \text{where } |p|+|q| \leq k \end{array},$$

where $g_{i\overline{j}}\left(l\right)$ denotes the components of the metric respectively to the coordinates on B_l and $C^\alpha(B_l)$ is the Hölder norm on B_l.

Remark 2.3. One notices that every geodesic on M can be extended to infinite and therefore a metric with bounded geometry is necessarily complete.

Example 2.4. Let M be complete Kähler manifold such that its sectional curvature and the covariant derivative of its scalar curvature are bounded. Then M has bounded geometry of order $2+\frac{1}{2}$ [T-Y, Proposition 1.2].

Definition 2.5. A vector bundle E is said to be of bounded geometry $k+\alpha$ if for all trivialization of E over every quasi-coordinate open set U, the corresponding transition automorphism $f_{U\cap V}$ is C^k bounded for every intersection $U\cap V\neq\emptyset$, in the sense that

$$\left|\frac{\partial^{|p|+|q|}}{\partial z^p \partial \overline{z}^q}\tilde{f}_{U\cap V}\right|_{C^\alpha(U\cap V)} \leq C_{|p|+|q|} \qquad \text{where } |p|+|q|\leq k,$$

where $\tilde{f}_{U\cap V}$ is the pull-back of $f_{U\cap V}$ on $U\cap V$, and $C^\alpha(U\cap V)$ is the Hölder norm on $U\cap V$.

Notation.

- The matrix $\left(g^{\overline{j}i}\right)$ denotes the inverse matrix of $\left(g_{i\overline{j}}\right)$, which means that we have $\sum_k g^{\overline{i}k}g_{k\overline{j}} = \sum_k g^{\overline{k}i}g_{\overline{k}j} = \delta_{ij}$. Eventually, we will associate to g, the normalized volume form $dV_g = dV = \frac{\omega^n}{n!}$.
- From now on, $\mathbf{O}(\alpha)$ will denote the Landau notation extended to the square matrices for which each entry can be bounded by $O(\alpha)$.
- We associate to a hermitian holomorphic vector bundle E equipped with a local frame (e_i) and to a metric h_E on that vector bundle, a natural hermitian matrix $\mathbf{h}_E$ with analytic coefficients defined by
$$(\mathbf{h}_E)_{ij} = h_E(e_i, e_j).$$
- Moreover $|.|_h$ denotes the pointwise norm associated to the metric h and $\|.\|_h$ the L^2 norm associated to $|.|_h$, i.e. $\left(\int_M |.|_h^2\, dV\right)^{1/2}$.

It is well known that there exists locally many Kähler potentials ψ for a Kähler metric ω,

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\psi.$$

We introduce a preferred Kähler potential using the work of Bochner [Ru, §3], [Boc1] (see also [C-G-R]) :

2.1. Canonical coordinates for a manifold and for a vector bundle

Proposition 2.6. *Suppose the metric ω is real analytic. For all point $z_0\in M$, there exists a unique change of coordinates (modulo affine transformation) for which there exists a Kähler potential $K_{z_0}(z)$ on (M,ω_g) which has locally the following Taylor expansion :*

$$K_{z_0}(z) = |z|^2 - \frac{1}{4}R_{i\overline{j}k\overline{l}}z_i\overline{z}_j z_k\overline{z}_l + O(|z|^5).$$

where z is the new coordinate after the change of variable and is called K-coordinate, and where $R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \sum_{p,q} g^{\overline{q}p} \frac{\partial g_{i\overline{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}$ is the full curvature tensor of the metric $g_{i\bar{j}}$.

Remark 2.7. The Taylor expansion of K_{z_0} may not converge if one does not assume that the metric is real analytic. In fact, we will only need a truncated Taylor expansion of K_{z_0} up to a certain order. This order will depend on the order of the asymptotic expansion of the generalized Bergman kernel we are looking for.
Therefore, as the truncated Taylor expansion of K_{z_0} always exists, the analyticity assumption will not be necessary.

Proposition 2.8. *Under the conditions of the previous proposition, there exists a holomorphic frame $(e_i)_{i=1..r}$ over a neighborhood of $z_0 \in M$ such that, with respect to this frame, the endomorphism associated to h_E has the expansion :*

$$\left(\mathbf{h}_E(z)\right)_{ij} = \left(\delta_{ij} - \sum_{1\leq k,l\leq n} \Theta(E)_{i\bar{j}k\bar{l}} z_k \bar{z}_l + \mathbf{O}\left(|z|^3\right)\right), \tag{2.1}$$

for all $1 \leq i, j \leq r$.

Proof. See [De2, Chapter V - Theorem 12.10].

2.2. L^2 estimates for the $\overline{\partial}$-operator

Let us note L_ω the contraction operator of $(1,1)$ type associated to the Kähler metric ω,

$$L_\omega u = \omega \wedge u$$

and $\Lambda_\omega := L_\omega^*$ the adjoint operator (that we will denote Λ for simplicity). Remember that the Bochner-Kodaira-Nakano identity asserts that for any Kähler manifold and any hermitian holomorphic vector bundle $\underline{E}$,

$$\Delta'' = \Delta' + [i\Theta(\underline{E}), \Lambda]$$

where $\Theta(\underline{E})$ is the curvature of $\underline{E}$, and Δ', Δ'' are Laplace-Beltrami operators associated to the Chern connexion $\nabla = D^{(1,0)} + D^{(0,1)}$ on $\underline{E}$:

$$\begin{aligned} \Delta' &= D^{(1,0)} D^{(1,0)*} + D^{(1,0)*} D^{(1,0)} \\ \Delta'' &= D^{(0,1)} D^{(0,1)*} + D^{(0,1)*} D^{(0,1)} \end{aligned}$$

By $Ric\left(\omega_g\right)$ we denote the Ricci curvature of the metric $\omega = \omega_g$. We know from the work of L. Hörmander a resolution of the "$\overline{\partial}$ problem" in the complete kählerian case, that has been precised by J-P. Demailly [De1] and which is the key point of the construction of peak sections.

Proposition 2.9. *Let M and L be chosen as previously, and $\underline{E}$ a hermitian holomorphic vector bundle of any rank for which the curvature operator $[i\Theta(\underline{E}), \Lambda]$ is definite positive and φ a L^1 function on M, limit of a decreasing sequence of smooth functions φ_j. If one has pointwisely for any j,*

$$\left\langle \frac{i}{2\pi} \partial\overline{\partial}\varphi_j + i\Theta\left(L\right) + Ric\left(\omega\right), v \wedge \overline{v} \right\rangle_g \geq C \left|v\right|_g^2 \tag{2.2}$$

for any tangent vector v of $(1,0)$-type, where $C>0$ denotes a constant that does not depend on j, then for any $(0,1)$-form w with values in $\underline{E}\otimes L$ on M satisfying

$$\begin{aligned}\overline{\partial} w &= 0,\\ \int_M |w|^2 e^{-\varphi} dV_{\underline{g}} &< \infty,\end{aligned}$$

there exists a smooth function u with values in $\underline{E}\otimes L$, such that

$$\overline{\partial} u = w,$$

and

$$\int_M |u|^2 e^{-\varphi} dV_g \leq \frac{2}{C}\int_M |w|^2 e^{-\varphi} dV_g,$$

where $|.|$ means the natural norm induced by h_L and the metric on $\underline{E}$.

Proof. First of all, from a general point of view,

$$\left\langle \left[i\Theta(\underline{E}\otimes\underline{E}'),\Lambda\right]u,u\right\rangle = \left\langle [i\Theta(\underline{E}),\Lambda]\,u,u\right\rangle + \left\langle \left[i\Theta(\underline{E}'),\Lambda\right]u,u\right\rangle.$$

However, by assumption we have ω is the curvature of the L which is positive, and considering L equipped with the metric $h_L e^{-\varphi_j}$, one can apply Theorem 8.4 of [De1] to the vector bundles $\underline{E}$ and $L\otimes K_M^{-1}$ with (2.2). We take the limit and use a monotony argument to conclude.

2.3. Peak Sections In that section, we follow W-D. Ruan [Ru] technique. From now on, we choose a local K-coordinates system such that at $z_0\in M$, the hermitian metric $\left(g_{i\overline{j}}\right)$ satisfies the conditions :

$$\begin{aligned} g_{i\overline{j}}(z_0) &= \delta_{ij},\\ \frac{\partial^{p_1+\ldots+p_n} g_{i\overline{j}}}{\partial z_1^{p_1}\ldots\partial z_n^{p_n}}(z_0) &= 0,\end{aligned}$$

for any n-tuple $P=(p_1,\ldots,p_n)\in\mathbb{Z}_+^n$ where $|P|:=p_1+\ldots+p_n\neq 0$. Moreover we assume that the hermitian metric h_E on the vector bundle E satisfies the identity (2.1) of Proposition 2.8, for a frame $(e_i)_{i=1..r}$ over a neighborhood of z_0. Let ϵ be a canonical section for the vector bundle (L,h_L) satisfying in a neighborhood of z_0, $|\epsilon|_{h_L}^2 = e^{-K_{z_0}}$. Under this setting and the assumptions of the main theorem, we construct sections whose L^2-norms have a Gaussian form around z_0 and for which we control completely the Taylor expansion at z_0. This gives an analogue of [Bou2, Theorem 3.1].

Proposition 2.10. *Fix $p'>0$ be an integer. For any n-tuple P of $\mathbb{Z}_+^n$ such that $p'>|P|$, there exists $k_0(p',M,\omega)>0$ such that for all $k>k_0$, there exists a family of global holomorphic sections $s_{k;i}^P := s_{p',P,k;i}$ of $H^0\left(M,E\otimes L^k\right)$ satisfying*

$$\left\|s_{k;i}^P\right\|_{h_k}^2 = \int_M \left|s_{k;i}^P\right|_{h_k}^2 dV = 1,$$

$$\int_{M\setminus\left\{|z|\leq\frac{\log k}{\sqrt{k}}\right\}} \left|s_{k;i}^P\right|_{h_k}^2 dV = O\left(\frac{1}{k^{2p'}}\right),$$

for $1 \leq i \leq r$, and one has the decomposition of $s^P_{k;i}$

$$s^P_{k;i}\left(z\right) = \widetilde{\upsilon}_{k,P,p';i}(z) + \upsilon_{k,P,p';i}(z)$$

where υ and $\widetilde{\upsilon}$ (which are not necessarily continuous) satisfy

$$\begin{aligned} \widetilde{\upsilon}_{k,P,p';i}(z) &= \begin{cases} \lambda_{P,k;i}\left(z^P + O(|z|^{2p'})\right)\left(1 + O\left(\frac{1}{k^{2p'}}\right)\right)\epsilon^k \otimes e_i \text{ if } |z| \leq \frac{\log k}{\sqrt{k}} \\ 0 \text{ if } |z| > \frac{\log k}{\sqrt{k}} \end{cases}, \\ \upsilon_{k,P,p';i}(z) &= O\left(|z|^{2p'}\right), \qquad \|\upsilon_{k,P,p';i}\|^2_{h_k} = O\left(1/k^{2p'}\right), \end{aligned}$$

where one has set $z^P = z_1^{p_1}...z_n^{p_n}$, and the coefficient $\lambda_{P,k;i}$ is completely determined by the geometry

$$\left(\lambda_{P,k;i}\right)^{-2} = \int_{\left\{|z| \leq \frac{\log k}{\sqrt{k}}\right\}} |z^P|^2 e^{-kK_{z_0}(z)} \left(\mathbf{h}_E\right)_{ii} dV.$$

Proof. Choose a cut-off function η such that $\eta(t) = 1$ for all $t < 1/2$, $\eta(t) = 0$ if $t > 1$, $0 \leq -\eta'(t) \leq 4$ and $|\eta''(t)| \leq 8$. We define the following weight function

$$\varphi(z) = (n + 2p')\, \eta\left(\frac{k|z|^2}{\left(\log k\right)^2}\right) \log|z|^2 \in L^1(M).$$

Then, we have

$$\partial\overline{\partial}\varphi = (1 + 2p')\left(\begin{array}{c} \left[\begin{array}{c} \eta''\left(\frac{k|z|^2}{(\log k)^2}\right)\frac{k^2}{(\log k)^4}\partial|z|^2 \wedge \overline{\partial}|z|^2 \\ +\eta'\left(\frac{k|z|^2}{(\log k)^2}\right)\frac{k}{(\log k)^2}\partial\overline{\partial}|z|^2 \end{array}\right]\log|z|^2 \\ +2\mathrm{Re}\left(\eta'\left(\frac{k|z|^2}{(\log k)^2}\right)\frac{k}{(\log k)^2}\partial|z|^2 \wedge \overline{\partial}\log|z|^2\right) \\ +\eta\left(\frac{k|z|^2}{(\log k)^2}\right)\partial\overline{\partial}\log|z|^2 \end{array}\right). \tag{2.3}$$

But $\eta''\left(\frac{k|z|^2}{(\log k)^2}\right) < 0$ or $\eta'\left(\frac{k|z|^2}{(\log k)^2}\right) < 0$ if and only if $\frac{1}{2} \leq \left(\frac{k|z|^2}{(\log k)^2}\right) \leq 1$, which is equivalent to

$$\frac{\left(\log k\right)^2}{2k} \leq |z|^2 \leq \frac{\left(\log k\right)^2}{k}.$$

For k large enough, $\frac{i}{2\pi}\partial\overline{\partial}|z|^2 \geq 0$ for $|z|^2 \leq \frac{(\log k)^2}{k}$. We obtain from (2.3) that

$$\frac{i}{2\pi}\partial\overline{\partial}\varphi \geq -\frac{kC\left(n + 2p'\right)}{\left(\log k\right)^2}\omega,$$

where C is a constant independent of k. This implies that for any unitary vector v of $(1,0)$-type, we have pointwisely on (M, ω_g)

$$\left\langle \frac{i}{2\pi}\partial\overline{\partial}\varphi_i + ki\Theta\left(L\right) + Ric\left(\omega\right), v \wedge \overline{v} \right\rangle_g \geq k\left(1 - \frac{C\left(n + 2p'\right)}{\left(\log k\right)^2}\right)|v|^2_g,$$

where we have used our assumption on the Ricci curvature of ω. For $i = 1, .., r$, we now set $\alpha = \frac{1}{4}\overline{\partial}\left(\eta\left(\frac{k|z|^2}{(\log k)^2}\right)z^P\epsilon^k \otimes e_i\right)$ and apply the Proposition 2.9 with $\underline{E} = E \otimes L^{k'}$ and

$k > k'$ for k' large enough. Hence, we obtain a section β with values in $E \otimes L^k$ such that $\overline{\partial}\beta = \alpha$ and

$$\int_M |\beta|^2_{h_k} e^{-\varphi} dV \leq \frac{2}{k\left(1 - \frac{C(n+2p')}{(\log k)^2}\right)} \int_M |\alpha|^2_{h_k} e^{-\varphi} dV,$$

for k large enough, we notice that

$$\frac{C\,(n+2p')}{(\log k)^2} \leq \frac{1}{2}.$$

Since $(\mathbf{h}_E)_{ij} = \left(\delta_{ij} - \sum_{1\leq k,l\leq n} \Theta(E)_{i\bar{j}k\bar{l}} z_k \bar{z}_l + \mathbf{O}\left(|z|^3\right)\right)$, if we define

$$Y_i\,(z) := 1 - \sum_{1\leq k,l\leq n} \Theta(E)_{i\bar{i}k\bar{l}} z_k \bar{z}_l + O\left(|z|^3\right),$$

this gives with C_0, C_1 independent constants,

$$\begin{aligned} \int_M |\beta|^2_{h_k} e^{-\varphi} dV &\leq \frac{C_0}{k} \int_M \left|\eta'\left(\frac{k|z|^2}{4\,(\log k)^2}\right)\right|^2 g^{\bar{j}i} z_i z_{\bar{j}} \frac{k^2}{(\log k)^4} |z^P|^2 e^{-kK_{z_0}(z)} Y_i\,(z)\, e^{-\varphi} dV \\ &\leq \frac{C_1}{(\log k)^2} \int_{\frac{2(\log k)^2}{k} \leq |z|^2 \leq \frac{4(\log k)^2}{k}} |z^P|^2 e^{-kK_{z_0}(z)} Y_i\,(z)\, e^{-\varphi} dV, \end{aligned} \tag{2.4}$$

because M and E have bounded geometry. From its definition, $\varphi(z) = 0$ if $|z|^2 \geq \frac{(\log k)^2}{k}$ and since

$$e^{-K_{z_0}(z)} = e^{-|z|^2 + O(|z|^4)},$$

and M has bounded geometry of order 2, we obtain the following upper bound for k large enough,

$$\begin{aligned} \int_M |\beta|^2_{h_k} e^{-\varphi} dV &\leq \frac{C_1}{(\log k)^2} \int_{\frac{2(\log k)^2}{k} \leq |z|^2 \leq \frac{4(\log k)^2}{k}} |z^P|^2 e^{-k|z|^2} dV, \\ &\leq \frac{C_1}{(\log k)^2} \left(\frac{4\,(\log k)^2}{k}\right)^{1+p} e^{-k\left(\frac{2(\log k)^2}{k}\right)}, \\ &< \frac{C'\,(\log k)^{2p}}{k^{1+p}} e^{-2(\log k)^2}. \end{aligned} \tag{2.5}$$

Now, for k large enough,

$$\frac{(\log k)^{2p}}{k^{1+p}} \leq \frac{1}{k^p}.$$

Hence, we have

$$\int_M |\beta|^2_{h_k} e^{-\varphi} dV = O\left(\frac{1}{k^p} e^{-2(\log k)^2}\right) = O\left(\frac{1}{k^{2p'}}\right).$$

We set $\widetilde{\beta}(z) = \eta\left(\frac{k|z|^2}{(\log k)^2}\right) z^P \epsilon^k \otimes e_i - \beta(z)$. We remark that $\overline{\partial}\widetilde{\beta} = 0$, i.e. $\widetilde{\beta}$ is a holomorphic section of $E \otimes L^k$, and this leads to

$$\begin{aligned} +\infty > \int_{\left\{|z|^2 < \frac{(\log k)^2}{2k}\right\}} |\beta|^2_{h_k} e^{-\varphi} dV &= \int_{\left\{|z|^2 < \frac{(\log k)^2}{2k}\right\}} |\beta|^2_{h_k} e^{-\left(n+2p'\right)\eta\left(\frac{k|z|^2}{(\log k)^2}\right)\log|z|^2} dV, \\ &= \int_M \frac{|\beta|^2_{h_k}}{|z^2|^{(n+2p')}} dV. \end{aligned}$$

Therefore we have at z_0 for $|z|^2 < \frac{(\log k)^2}{2k}$ that $u(z) = O\left(|z|^{2p'}\right)$, and moreover over a neighborhood of z_0,

$$\widetilde{\beta}(z) = z^P \epsilon^k \otimes e_i + O\left(|z|^{2p'}\right).$$

The same arguments as for (2.5) imply that

$$\begin{aligned}
\int_M |\widetilde{\beta}|^2_{h_k} dV &= \int_M \left|\eta\left(\frac{k|z|^2}{(\log k)^2}\right)\right|^2 |z^P|^2 e^{-kK_{z_0}(z)} Y_i(z)\, dV \\
&+2\mathrm{Re}\left(\int_M \left\langle \eta\left(\frac{k|z|^2}{(\log k)^2}\right)^2 z^P \epsilon^k \otimes e_i, \beta\right\rangle_{h_k} dV\right) + \int_M |\beta|^2_{h_k} dV, \\
&= \int_{\left\{\frac{k|z|^2}{(\log k)^2} \leq 1\right\}} |z^P|^2 e^{-kK_{z_0}(z)} Y_i(z)\, dV + O\left(\frac{1}{k^p} e^{-2(\log k)^2}\right).
\end{aligned}$$

Eventually, we conclude by defining the following section :

$$s^P_{k;i}(z) = \frac{\widetilde{\beta}}{\left\|\widetilde{\beta}\right\|_{h_k}}.$$

We will need the following result [Ru, lemme 3.3]:

Lemma 2.11. *On $\mathbb{C}^n$ the following identity holds :*

$$\int_{|z| \leq \frac{\log k}{\sqrt{k}}} |z_1^{p_1} \ldots z_n^{p_n}|^2 e^{-k|z|^2} dV = \left(\frac{\pi}{k}\right)^n \frac{\prod_{i=1}^n (p_i!)}{k^p} + O\left(\frac{1}{k^{2p'}}\right),$$

where $p := \sum p_i$, $p' > p$.

Proposition 2.12. *Let T be another holomorphic section of $E \otimes L^k$. Then, with the same notations as before,*

1. *if z^P does not belong to the Taylor expansion of T at z_0, there exists a constant C_1 that depends only on the geometry of the manifold and independent of k such that*
$$\int_M \left\langle s^P_{k;i}, T\right\rangle_{h_k} dV \leq \frac{C_1}{k} \left|\left|T\right|\right|_{h_k},$$
2. *if the Taylor expansion of T at z_0 does not contain any term of the form z^Q such that $|Q| < |P| + d$ (where d integer such that $d \geq 1$ and $d \leq p' - n - p$), there exists a constant C_2 that depends only on the geometry of the manifold and independent of k such that*
$$\int_M \left\langle s^P_{k;i}, T\right\rangle_{h_k} dV \leq \frac{C_2}{k^{1+d/2}} \left|\left|T\right|\right|_{h_k}.$$

Proof. We focus on point **2.**, case **1.** is similar. One can write T as

$$T = \sum_{i=1}^r t_i \epsilon^k \otimes s^P_{k;i}.$$

and its Taylor expansion is given by the Taylor expansion of each t_i in the system of coordinates $(z_1, ..., z_n)$. Set

$$\sum_{1\le k_1,k_2\le n} a_{k_1\overline{k_2}} z_{k_1}\bar{z}_{k_2} := e^{k\left(|z|^2 - K_{z_0}(z)\right)} (\mathbf{h}_E)_{ii} \det\left(g_{i\bar{j}}\right).$$

Therefore, since $s_{k;i}^P(z) = \lambda_{P,k;i}\left(1 + O\left(\frac{1}{k^{2p'}}\right)\right)\left(z^P + O\left(|z|^{2p'}\right)\right)\epsilon^k \otimes e_i$ if $|z| \le \frac{\log k}{\sqrt{k}}$, and by setting $d\mu = \bigwedge_{i=1}^n dz_i \wedge d\bar{z}_i$,

$$\begin{aligned}
\frac{1}{\lambda_{P,k;i}} \int_M \left\langle s_{k;i}^P, T\right\rangle_{h_k} dV &= \int_{|z|\le \frac{\log k}{\sqrt{k}}} z^P \overline{t_i(z)} e^{-k|z|^2} \left(\sum_{1\le k_1,k_2\le n} a_{k_1\overline{k_2}} z_{k_1}\bar{z}_{k_2}\right) d\mu \\
&\quad + O\left(\frac{1}{k^{p-n}} e^{-2(\log k)^2}\right) ||T||_{h_k}, \\
&= \int_{|z|\le \frac{\log k}{\sqrt{k}}} z^P \overline{t_i(z)} e^{-k|z|^2} \left(\sum_{k_1-k_2<d} a_{k_1\overline{k_2}} z_{k_1}\bar{z}_{k_2}\right) d\mu \\
&\quad + \int_{|z|\le \frac{\log k}{\sqrt{k}}} z^P \overline{t_i(z)} \left(O\left(k|z|^{d+4} + |z|^{d+2}\right)\right) e^{-kK_{z_0}(z)} (\mathbf{h}_E)_{ii}\, dV \\
&\quad + O\left(\frac{1}{k^{p-n}} e^{-2(\log k)^2}\right) ||T||_{h_k}, \\
&\le C\left(\int_{|z|\le \frac{\log k}{\sqrt{k}}} |z^p|^2 \left(k^2|z|^{2d+8} + |z|^{2d+4}\right) e^{-kK_{z_0}(z)} (\mathbf{h}_E)_{ii}\, dV\right)^{1/2} ||T||_{h_k} \\
&\quad + O\left(\frac{1}{k^{p-n}} e^{-2(\log k)^2}\right) ||T||_{h_k},
\end{aligned}$$

where we have used the boundedness geometry assumption. By Lemma 2.11, we have

$$\int_M \left\langle s_{k;i}^P, T\right\rangle_{h_k} dV = O\left(\frac{1}{k^{\frac{d+2}{2}}}\right) \left\| s_{k;i}^P \right\|_{h_k} ||T||_{h_k}.$$

Remark 2.13. With same notations as previously, for every $P, Q \in \mathbb{Z}_+^n$,

$$\int_M \left\langle s_{p',P,k;i}, s_{p',Q,k;i}\right\rangle_{h_k} dV = \left(\frac{\pi}{k}\right)^n \sum_{\alpha+|P|=\beta+|Q|} \mathsf{a}_{\alpha\overline{\beta}} \frac{(\alpha+|P|)!}{k^{\alpha+|P|}},$$

where $\mathsf{a}_{\alpha\overline{\beta}}$ are polynomials expressions of the curvature of ω and of h_E and their covariant derivatives. These coefficients are explicitly given by a finite number of algebraic operations.

3. Proof of Theorem 1.1

We shall begin this section by giving some technical lemmas before finishing the proof of Theorem 1.1.

Definition 3.1. Let $p, n > 0$ be integers. A function f on $\{1, ..., n\}^p \times \{1, ..., n\}^p$ is said to be symmetric if

$$f(\alpha(I), \beta(I)) = f(I, J),$$

where $I, J \in \{1, ..., n\}^p \times \{1, ..., n\}^p$ for any permutations $\alpha, \beta \in \mathbb{S}_n$.

Remark 3.2. The curvature tensor $R_{i\bar{j}k\bar{l}}$ and the Ricci curvature tensor are examples of symmetric functions.

Lemma 3.3. *Let $q > 0$ be an integer. If f is symmetric on $\{1, ..., n\}^p \times \{1, ..., n\}^p$, then for any $p' > p$,*

$$\begin{aligned}
&\sum_{(i,j)} \int_{|z| \leq \frac{\log k}{\sqrt{k}}} f(I, J) z_{i_1} ... z_{i_p} \bar{z}_{j_1} ... \bar{z}_{j_p} |z|^{2q} e^{-k|z|^2} dV \\
= &\left(\sum_i f(I, J) \right) \frac{p!\,(n+p+q-1)!}{(p+n-1)!k^{n+p+q}} + O\left(\frac{1}{k^{n+q+p'}} \right),
\end{aligned}$$

where $I = (i_1, ..., i_p)$ and $J = (j_1, ..., j_p)$ are p-tuples, with $1 \leq i_k \leq n$, $1 \leq j_k \leq n$ for $k \in \{1, .., p\}$.

Proof. See [Lu, Lemma 4.1].

Notation. The family $s_{k;i}^P$ defined by Proposition 2.10 is a basis of a supplement in $H^0(M, E \otimes L^k)$ of the kernel of the surjective linear application $H^0(M, E \otimes L^k) \to \mathcal{O}_M(E \otimes L^k)/\mathfrak{m}_{z_0}^{p'}$ that we shall denote $\mathcal{K}_{k,p',z_0}$:

$$\mathcal{K}_{k,p',z_0} = \left\{ S \in H^0(M, E \otimes L^k) : \left(\nabla_E \otimes \nabla_L^k\right)^{P'} (S)(z_0) = 0 \text{ for any } |P'| \leq p' \right\},$$

where $\nabla_E \otimes \nabla_L^k$ is the covariant derivative on $E \otimes L^k$ associated to the Chern connexion and $P' \in \mathbb{Z}_+^n$. Roughly speaking, $\mathcal{K}_{k,p',z_0}$ will parametrize the space of sections that will not be considered to compute the asymptotic expansion of the generalized Bergman kernel.

Fix now $p' = 1$, $P = \mathbf{0} := (0, ..., 0) \in \mathbb{Z}_+^n$ and set

$$\boldsymbol{S}_i := \frac{1}{\lambda_{\mathbf{0},k;i}} s_{1,\mathbf{0},k;i}$$

by applying Proposition 2.10. We are going to define a hilbertian basis of holomorphic sections at the point $z_0 \in M$. Of course $\{\boldsymbol{S}_i : 1 \leq i \leq r\}$ and $\mathcal{K}_{k,1,z_0}$ generate $H^0(M, E \otimes L^k)$ for k large enough. Let $(\boldsymbol{T}_j)_j$ be a hilbertian basis of $\mathcal{K}_{k,1,z_0}$. The $\boldsymbol{T}_j$ sections vanish at z_0, and can be chosen such that for $1 \leq i \leq r$ and $j > r$, one has $\int_M \langle \boldsymbol{S}_i, \boldsymbol{T}_j \rangle dV = 0$. Set

$$\widehat{s}_i := \begin{cases} \boldsymbol{S}_i & r \geq i \geq 1 \\ \boldsymbol{T}_{i-r} & i \geq r+1 \end{cases} \tag{3.1}$$

with always orthonormal $(\boldsymbol{T}_j)_{j>r+1}$ vanishing at z_0. Let $\mathsf{B} \in \mathbb{M}_{2r \times 2r}(\mathbb{C})$ be the matrix whose entries are

$$\mathsf{B}_{ij} = \int_M \langle \widehat{s}_i, \widehat{s}_j \rangle_{h_k} dV \ , 1 \leq i, j \leq 2r. \tag{3.2}$$

The matrix B_{ij} is hermitian definite positive and consequently, one can write

$$\mathsf{B}_{ij}=\sum_{k=1}^{2r}\mathsf{C}_{ik}\overline{\mathsf{C}_{jk}},$$

where C is an inversible matrix $2r\times 2r$. Denote $\mathring{\mathsf{C}}$ the inverse matrix C. We check that the family $\left\{\left(S_i:=\sum_{k=1}^{2r}\mathring{\mathsf{C}}_{ik}\widehat{s_k}\right)_{i=1..r},(T_j)_{j>r}\right\}$ is a hilbertian basis of $H^0(M,E\otimes L^k)$ for $\int_M\langle .,.\rangle_{h_k}dV$. Therefore, if $\left(\mathbf{e}_i=e_i\otimes\epsilon^k\right)_{i=1..r}$ is a local frame of $E\otimes L^k$ (for h_k) over a neighborhood of $z_0\in M$, the generalized Bergman kernel $\widehat{\mathsf{B}}$ defined by (1.2) satisfies at that point for $1\le i,j\le r$,

$$\begin{aligned}\left\langle\widehat{\mathsf{B}}\mathbf{e}_i,\mathbf{e}_j\right\rangle_{h_k}(z_0) &= \sum_{k=1}^{2r}S_{kj}\overline{S_{ki}},\\ &= \sum_{k=1}^{2r}\left(\sum_{l=1}^{2r}\mathring{\mathsf{C}}_{kl}S_{lj}\right)\overline{\left(\sum_{m=1}^{2r}\mathring{\mathsf{C}}_{km}S_{mi}\right)},\\ &= \sum_{k=1}^{2r}\mathring{\mathsf{C}}_{kj}\overline{\mathring{\mathsf{C}}_{ki}}=\mathsf{B}^{-1}_{ij}.\end{aligned}$$

where one has used the decomposition $S_i(z_0)=\sum S_{ij}\mathbf{e}_j(z_0)$ and the fact that we have at z_0, $\langle\widehat{s_i}(z_0),\mathbf{e}_j(z_0)\rangle_{h_k}=\delta_{ij}$ for $r\ge i,j\ge 1$. This relation enables us to express the generalized Bergman kernel using the peak sections that we have just defined.

With this setting, let us introduce the $r\times r$ blocks decomposition of B :

- $\mathsf{B}_{1,1}=\left(\int_M\langle\boldsymbol{S}_i,\boldsymbol{S}_j\rangle_{h_k}dV\right)_{1\le i,j\le r}$
- $\mathsf{B}_{1,2}=\left(\int_M\langle\boldsymbol{S}_i,\boldsymbol{T}_j\rangle_{h_k}dV\right)_{1\le i,j\le r}$
- $\mathsf{B}_{2,1}=\left(\int_M\langle\boldsymbol{T}_i,\boldsymbol{S}_j\rangle_{h_k}dV\right)_{1\le i,j\le r}$
- $\mathsf{B}_{2,2}=Id_{r\times r}$.

Let $\mathring{\mathsf{B}}\in\mathbb{M}_{2r\times 2r}(\mathbb{C})$ be the matrix with the $r\times r$ block decomposition :

$$\mathring{\mathsf{B}}=\begin{pmatrix}\left(\mathsf{B}_{1,1}-\mathsf{B}_{1,2}\,{}^t\overline{\mathsf{B}_{1,2}}\right)^{-1} & -\left(\mathsf{B}_{1,1}-\mathsf{B}_{1,2}\,{}^t\overline{\mathsf{B}_{1,2}}\right)^{-1}\mathsf{B}_{1,2}\\ -\left(\mathsf{B}_{1,1}-\mathsf{B}_{1,2}\,{}^t\overline{\mathsf{B}_{1,2}}\right)^{-1}\,{}^t\overline{\mathsf{B}_{1,2}} & \left(\mathsf{B}_{1,1}-\mathsf{B}_{1,2}\,{}^t\overline{\mathsf{B}_{1,2}}\right)^{-1}\mathsf{B}_{1,1}\end{pmatrix}.$$

Setting $\mathsf{U}=\left(\mathsf{B}_{1,1}-\mathsf{B}_{1,2}\,{}^t\overline{\mathsf{B}_{1,2}}\right)^{-1}\left(\mathsf{B}_{1,1}\,{}^t\overline{\mathsf{B}_{1,2}}-{}^t\overline{\mathsf{B}_{1,2}}\mathsf{B}_{1,1}\right)$, one notices that

$$\mathring{\mathsf{B}}=\left(Id+\begin{pmatrix}0&0\\ \mathsf{U}&0\end{pmatrix}\right)\mathsf{B}^{-1}.$$

With Proposition 2.12 ((1) and (2) with $d=1$), we directly obtain that $\mathsf{B}_{1,2}=\mathbf{O}\left(\frac{1}{k^{3/2}}\right)$ and also that $\mathsf{U}=\mathbf{O}\left(\frac{1}{k^{3/2}}\right)$. Consequently, the first $r\times r$ block of $\mathring{\mathsf{B}}$ is

$$\begin{aligned}\mathring{\mathsf{B}}_{1,1} &= \left(\mathsf{B}^{-1}\right)_{1,1}\left(Id+\mathbf{O}\left(\frac{1}{k^3}\right)\right)\left(Id+\mathbf{O}\left(\frac{1}{k^{3/2}}\right)\right),\\ &= \left(\mathsf{B}^{-1}\right)_{1,1}\left(Id+\mathbf{O}\left(\frac{1}{k^{3/2}}\right)\right).\end{aligned}\tag{3.3}$$

With Proposition 2.10, we know the entries of the matrix $\mathsf{B}_{1,1}$:

$$\int_M \langle \boldsymbol{S}_i, \boldsymbol{S}_j \rangle_{h_k} dV = \int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} e^{-kK_{z_0}(z)} \left(\mathbf{h}_E\right)_{ij} dV + \mathbf{O}\left(\frac{1}{k^{n+2}}\right). \tag{3.4}$$

At that point, we use the fact that with the $K-$coordinates at z_0,

$$\det\left(g_{i\bar{j}}\right) = e^{-Ric_{i\bar{j}} z_i \bar{z}_j + O\left(|z|^3\right)},$$

where $Ric\,(g)_{i\bar{j}} = Ric_{i\bar{j}} := \sum_{l,k} g^{\bar{l}k} R_{i\bar{l}k\bar{j}}$. Since we also have $d\mu = \bigwedge_{i=1}^{n} dz_i \wedge d\bar{z}_i$, we immediately get

$$\begin{aligned}
\mathsf{B}_{1,1} &= \int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} \left(e^{-k|z|^2 + \frac{k}{4} R_{i\bar{j}k\bar{l}} z_i \bar{z}_j z_k \bar{z}_l + O\left(|z^5|\right)}\right) Id_{r\times r} \times \\
&\quad \left(Id_{r\times r} - \Theta(E)_{i\bar{j}k\bar{l}} z_k \bar{z}_l + \mathbf{O}\left(|z|^3\right)\right) \det\left(g_{i\bar{j}}\right) d\mu + \mathbf{O}\left(\frac{1}{k^{n+2}}\right), \\
&= \int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} e^{-k|z|^2} \left(1 + \frac{k}{4} R_{i\bar{j}k\bar{l}} z_i \bar{z}_j z_k \bar{z}_l + O\left(|z^5|\right) + \sum_{j\geq 2} k^j \times O\left(|z^{4j}|\right)\right) \times \\
&\quad \left(Id_{r\times r} - \Theta(E)_{i\bar{j}k\bar{l}} z_k \bar{z}_l + \mathbf{O}\left(|z|^3\right)\right) \left(1 - Ric_{i\bar{j}} z_i \bar{z}_j + O\left(|z|^3\right)\right) d\mu \\
&\quad + \mathbf{O}\left(\frac{1}{k^{n+2}}\right), \\
&= \int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} e^{-k|z|^2} \left(Id_{r\times r}\left(1 - Ric_{i\bar{j}} z_i \bar{z}_j + \frac{k}{4} R_{i\bar{j}k\bar{l}} z_i \bar{z}_j z_k \bar{z}_l\right) - \Theta(E)_{i\bar{j}k\bar{l}} z_k \bar{z}_l\right) \\
&\quad + e^{-k|z|^2} \left(\gamma\left(|z^3|\right) + k\gamma'\left(|z^5|\right)\right) + e^{-k|z|^2} \mathbf{O}\left(|z|^5\right) d\mu + \sum_{j\geq 2} k^j \times \mathbf{O}\left(\frac{1}{k^{n+2j}}\right) \\
&\quad + \mathbf{O}\left(\frac{1}{k^{n+2}}\right).
\end{aligned}$$

But the scalar curvature of the metric ω_g is $Scal\,(g) = \sum_{i,j} g^{\bar{j}i} Ric_{i\bar{j}}$ and

$$\int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} e^{-k|z|^2} \left(\gamma\left(|z|^3\right) + k\gamma'\left(|z^5|\right)\right) d\mu = 0$$

and D. Catlin's expansion in [Ca] prescribes that all the terms in $k^{(2p-1/2)}$ vanish (these terms are annihilated by the asymmetry in the variables z and $\bar{z}$, see [Lu, p. 16]). Therefore, we get by Lemma 3.3,

$$\begin{aligned}
\mathsf{B}_{1,1} &= \frac{1}{k^n} Id_{r\times r} - \frac{Scal(g) Id_{r\times r}}{k^{n+1}} + \frac{2!}{k^{n+2}} \left(\frac{k}{4} Scal(g) Id_{r\times r}\right) - \frac{1}{k^{n+1}} \sqrt{-1}\Lambda_\omega \Theta\left(E\right) \\
&\quad + \mathbf{O}\left(\frac{1}{k^{n+2}}\right), \\
&= \frac{1}{k^n} Id_{r\times r} - \frac{1}{k^{n+1}} \left(\frac{1}{2} Scal(g) Id_{r\times r} + \sqrt{-1}\Lambda\Theta\left(E\right)\right) + \mathbf{O}\left(\frac{1}{k^{n+2}}\right).
\end{aligned}$$

Since we have already seen that the asymptotic expansion to order two of $\widehat{\mathsf{B}}(z_0)$ is reduced to the computation of $\left(\mathsf{B}^{-1}(z_0)\right)_{1\leq i,j\leq r}$ by (3.3), we have for the terms of weight k^{n-1} and k^n that:

$$\widehat{\mathsf{B}}(z_0) = \mathsf{a}_0(z_0)k^n + \mathsf{a}_1(z_0)k^{n-1} + ... \tag{3.5}$$

and clearly for $\mathsf{a}_i(E,L) \in End(E)$:

$$\mathsf{a}_0 = Id, \qquad \mathsf{a}_1 = \frac{1}{2}Scal(g)Id_{r\times r} + \sqrt{-1}\Lambda\Theta\left(E\right).$$

Note that in the computations, the bounds of the derivatives of the Ricci curvature are required to control the terms in $O(|z|^3)$ while 3rd order bounds are required on the metrics h_L and h_E. This leads to ask for boundedness of the geometry of order 5. This concludes the proof of our main theorem.

4. Generalization of Theorem 1.1

4.1. Case of an unspecified polarization Suppose L is a polarization on (M,ω_g). In the case where we do not make any assumption on the curvature of L, the same arguments can be applied considering the Kähler form $\omega_L = -\frac{i}{2\pi}\partial\overline{\partial}\log\left(h_L\right)$. Let's introduce the diastasis K^L respectively to ω_L. Then, if one denotes R^L the full curvature tensor of the metric ω_L, we get in a neighborhood of a point $z_0 \in M$, in a local K^L-coordinates system, $(z'_i)_{i=1..n}$,

$$K^L_{z_0}(z') = |z'|^2 - \frac{1}{4}R^L_{i\bar{j}k\bar{l}}z'_i\bar{z}'_j z'_k\bar{z}'_l + O(|z'|^5).$$

Moreover, fix $p'=1$, $P = \mathbf{0} := (0,...,0) \in \mathbb{Z}^n_+$. The proof of Proposition 2.10 can be immediately updated to build sections

$$\boldsymbol{S}_i := \frac{1}{\lambda^L_{\mathbf{0},k;i}} s^L_{1,\mathbf{0},k;i}$$

where orthonormal sections $s^L_{1,\mathbf{0},k;i}$ satisfy

$$\left\|s^L_{1,\mathbf{0},k;i}\right\|^2_{h_k} = 1, \qquad \int_{M\setminus\left\{|z'|\leq\frac{\log k}{\sqrt{k}}\right\}}\left|s^L_{1,\mathbf{0},k;i}\right|^2_{h_k} dV_g = O\left(\frac{1}{k^{2p'}}\right),$$

with the decomposition :

$$\begin{aligned}
s^L_{1,\mathbf{0},k;i}(z') &= \widetilde{\upsilon}^L_{k;i}(z') + \upsilon^L_{k;i}(z'),\\
\widetilde{\upsilon}^L_{k;i}(z') &= \begin{cases} \lambda^L_{\mathbf{0},k;i}\left(1+O(|z'|^2)\right)\left(1+O\left(\frac{1}{k^2}\right)\right)\epsilon^k\otimes e_i \text{ if } |z'|\leq\frac{\log k}{\sqrt{k}}\\ 0 \text{ if } |z'| > \frac{\log k}{\sqrt{k}}\end{cases}\\
\upsilon^L_{k;i}(z') &= O\left(|z'|^2\right), \qquad \left\|\upsilon^L_{k;i}\right\|^2_{h_k} = O\left(1/k^2\right),\\
\left(\lambda^L_{\mathbf{0},k;i}\right)^{-2} &= \int_{\left\{|z'|\leq\frac{\log k}{\sqrt{k}}\right\}} e^{-kK^L_{z_0}(z')}\left(\mathbf{h}_E\right)_{ii} dV_g.
\end{aligned}$$

Let $\left(\boldsymbol{T}_j\right)_j$ be a hilbertian basis of $\mathcal{K}_{k,1,z_0}$. The sections $\boldsymbol{T}_j$ can be chosen such that for $1\leq i\leq r$ and $j>r$, one has $\int_M \langle \boldsymbol{S}_i, \boldsymbol{T}_j\rangle dV = 0$. Set

$$\widehat{s}_i := \begin{cases} \boldsymbol{S}_i & r\geq i\geq 1\\ \boldsymbol{T}_{i-r} & 2r\geq i\geq r+1\end{cases}, \qquad \mathsf{B}_{ij} = \int_M \langle\widehat{s}_i,\widehat{s}_j\rangle_{h_k} dV \in \mathbb{M}_{2r\times 2r}\left(\mathbb{C}\right).$$

The generalized Bergman Kernel $\widehat{\mathsf{B}}$ defined by (1.2) is given by the computation of $\left(\mathsf{B}^{-1}(z_0)\right)_{1\leq i,j\leq r}$ where

$$\mathsf{B}(z_0) = \begin{pmatrix} \int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} |\epsilon|^2_{h_L}\mathbf{h}_E dV_g & \int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} \langle \boldsymbol{S}_i, \boldsymbol{T}_j\rangle_{h_k} dV_g \\ \int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} \langle \boldsymbol{T}_i, \boldsymbol{S}_j\rangle_{h_k} dV_g & Id_{r\times r}\end{pmatrix} + \mathbf{O}\left(\frac{1}{k^{n+2}}\right).$$

Therefore, by change of variable in K^L-coordinates, we have for the first $r\times r$ block of B :

$$\begin{aligned}\mathsf{B}_{1,1} &= \int_{\left\{|z'|\leq \frac{\log k}{\sqrt{k}}\right\}} \left(e^{-k|z'|^2+\frac{k}{4}R^L_{i\bar{j}k\bar{l}}z'_i\bar{z}'_j z'_k\bar{z}'_l+O\left(|z'^5|\right)}\right) Id_{r\times r}\times \\ &\quad \left(Id_{r\times r}-\Theta(E)_{i\bar{j}k\bar{l}}z'_k\bar{z}'_l+\mathbf{O}\left(|z'|^3\right)\right)\det\left(g_{i\bar{j}}\right) det_\omega(\omega_L)\, d\mu+\mathbf{O}\left(\frac{1}{k^{n+2}}\right) \\ &= \int_{\left\{|z|\leq \frac{\log k}{\sqrt{k}}\right\}} e^{-k|z'|^2} det_\omega(\omega_L)\begin{pmatrix} Id_{r\times r}\left(1-Ric(g)_{i\bar{j}}\,z'_i\bar{z}'_j+\frac{k}{4}R^L_{i\bar{j}k\bar{l}}z'_i\bar{z}'_j z'_k\bar{z}'_l\right) \\ -\Theta(E)_{i\bar{j}k\bar{l}}z'_k\bar{z}'_l\end{pmatrix} \\ &\quad +e^{-k|z'|^2}\left(\mathbf{O}\left(|z'|^3\right)+k\times\mathbf{O}\left(|z'|^5\right)\right)+e^{-k|z'|^2}\mathbf{O}\left(|z'|^5\right)d\mu+\mathbf{O}\left(\frac{1}{k^{n+2}}\right)\end{aligned}$$

with the same arguments as in the previous section. We obtain, using Lemma 3.3, at the point z_0 considered,

$$\begin{aligned}\mathsf{B}_{1,1} &= det_\omega(\omega_L)\begin{pmatrix}\frac{1}{k^n}Id_{r\times r}-\frac{\sum_{i,j} g_L^{\bar{j}i}Ric(g)_{i\bar{j}}Id_{r\times r}}{k^{n+1}}+\frac{2!}{k^{n+2}}\left(\frac{k}{4}Scal(g_L)Id_{r\times r}\right) \\ -\frac{1}{k^{n+1}}\sqrt{-1}\Lambda_\omega\Theta(E)\end{pmatrix} \\ &\quad +\mathbf{O}\left(\frac{1}{k^{n+2}}\right).\end{aligned}$$

Therefore we get :

Theorem 4.1. *Let (M,ω) be a complete Kähler manifold with bounded geometry of order $3+0$ and (E,h_E) a hermitian holomorphic vector bundle with bounded geometry of order $3+0$. Let (L,h_L) be a holomorphic line bundle on M such that its curvature is a definite positive form and with bounded geometry of order $3+0$. We have the following asymptotic expansion for the generalized Bergman kernel on $E\otimes L^k$ when $k\to\infty$:*

$$\left\|\begin{matrix}\widehat{\mathsf{B}}_k-k^n det_\omega(\omega_L)\,Id_{r\times r} \\ -k^{n-1}det_\omega(\omega_L)\left(\left(tr_{\omega_L}(Ric(g))-\frac{1}{2}Scal(g_L)\right)Id_{r\times r}+\sqrt{-1}\Lambda_{\omega_L}\Theta(E)\right)\end{matrix}\right\|_{C^0} = O(k^{n-2}).$$

where one has assumed that the smooth hermitian metrics h_E, h_L and the Kähler form ω and all their derivatives of order ≤ 3 belong to a compact set for the C^0 topology.

Remark 4.2. The theorem holds when one does not assume that ω is complete but that M is weakly pseudo-convex (i.e. one assumes the existence of a plurisubharmonic exhaustion function on M), see [De1, Section 8]. In particular, if $\bar{M}$ is a smooth compact Kähler manifold and D is an ample divisor, then the asymptotic expansion of the generalized Bergman kernel on $M:=\bar{M}\backslash D$ is given by the above formula.

4.2. How to compute higher order terms? We now outline the computation one needs to do in order to compute the asymptotic expansion of the generalized Bergman kernel of higher order.

- Computation of the a_2 term:
  The a_2 term can be obtained from the formula (3.4) provided that we get an expansion of higher order for $K_{z_0}(z)$, $\mathbf{h}_E$, $\det\left(g_{i\bar{j}}\right)$. In fact, one notices that $\mathring{\mathsf{B}}_{1,1} = \left(\mathsf{B}^{-1}\right)_{1,1}\left(Id + \mathbf{O}\left(\frac{1}{k^3}\right)\right)$ using a result a little bit more precise from Z. Lu [Lu, Proposition 3.1]. For the computation of $\mathring{\mathsf{B}}_{1,1}$, we need an asymptotic expansion to order 2 of $\lambda^2_{\mathbf{0},k;i}$ and of $\int_M \langle \widehat{s_i}, \widehat{s_j}\rangle_{h_k} dV$ of the following form
  $$\begin{aligned}\lambda^2_{\mathbf{0},k;i} &= k^n + k^{n-1}\alpha_i + k^{n-2}\beta_i + O\left(k^{n-3}\right),\\ \int_M \langle \widehat{s_i}, \widehat{s_j}\rangle_{h_k} dV &= \frac{\alpha'_{ij}}{k} + \frac{\beta'_{ij}}{k^2} + O\left(\frac{1}{k^3}\right) \qquad \text{for } i \neq j,\end{aligned}$$
  where $\alpha_i, \beta_i, \alpha'_{ij}, \beta'_{ij}$ depend only on ω, the metric on E and their covariant derivatives. Therefore, one needs the Taylor expansion of the diastasis, and of $\mathbf{h}_E$, till order 6. Eventually, we need a Taylor expansion to order 5 of $\det\left(g_{i\bar{j}}\right)$.
- Computation of the a_3 term:
  Compared with the computation of a_2, we need to find higher order expansion of the first $r \times r$ block of B:
  $$\mathring{\mathsf{B}}_{1,1} = \left(\mathsf{B}^{-1}\right)_{1,1} + \mathsf{Q}^t\overline{\mathsf{Q}} + \mathbf{O}\left(\frac{1}{k^{n+7/2}}\right)$$
  which is given considering a new basis $\widehat{s_i}$ formed this time, from $\{\boldsymbol{S}_i : 1 \leq i \leq r\} \cup \left\{\boldsymbol{S}_{i,P} := \frac{1}{\lambda_{P,k;i}} s_{2,P,k;i} : |P| = 1 \text{ and } 1 \leq i \leq r\right\}$ and from a hilbertian basis of $\mathcal{K}_{k,2,z_0}$, and for which $\mathsf{Q}_{ij} = \int_M \left\langle \boldsymbol{S}_i, \boldsymbol{S}_{i,P_j}\right\rangle dV$ with $P_j = (\delta_{1j}, \ldots, \delta_{nj}) \in \mathbb{Z}^n_+$ where we have used Kronecker symbol.
- The other terms a_i can be given by a similar way as in cases a_2 and a_3, by finding higher order asymptotic expansions of $K_{z_0}(z)$, $\mathbf{h}_E$, $\det\left(g_{i\bar{j}}\right)$ to a higher order and considering the spaces $\mathcal{K}_{k,p,z_0}$.

For example, in the case of a polarized manifold assuming that equation (1.1) holds, we find for the term of weight k^{n-2} of the asymptotic expansion of the generalized Bergman kernel the following expression.

$$\begin{aligned} a_2 \;=\; & \frac{1}{24}\left(|R_{i\bar{j}k\bar{l}}|^2 + 3Scal(g)^2 - 4|Ric(\omega)|^2 + 8\Delta Scal(g)\right) Id \\ & -\frac{1}{2}\left(\Lambda\Theta(E)\Lambda\Theta(E) + \Lambda\partial\bar{\partial}\Lambda\Theta(E) + \Theta(E)_{i\bar{j}}\Theta(E)_{i\bar{j}} + \Theta(E)_{j\bar{i}}Ric(\omega)_{i\bar{j}}\right) \\ & +\frac{1}{2}Scal(g)\sqrt{-1}\Lambda\Theta(E). \end{aligned}$$

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