Bergman kernel for sections vanishing along a divisor and slope stability

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1 Introduction

The celebrated Kobayashi-Hitchin correspondance asserts that a holomorphic vector bundle over a projective manifold is Mumford polystable if and only if it can be equipped with a Hermitian-Einstein metric on it. The "easy" sense of this correspondance is the implication *existence of a Hermitian-Einstein metric* $\Rightarrow$ *Mumford stability.* It has been proved in the Ph.D thesis of M. Lübke [Lub] and we refer to [LT, Th] as surveys on this correspondance and the notion of stabilities that we shall mention.

In the world of smooth projective manifolds, it is expected (Conjecture of Yau-Tian-Donaldson [Do1, Ya1, Ya2]) that a similar correspondence holds between K-stability and the existence of a constant scalar curvature metric. In [RT1, RT2], it is introduced a notion of slope stability (derived as a special case from the notion of K-stability) for a couple (M, L) where M is a manifold and L a polarization. We expect that a proof of the "easy" sense of the correspondance could be given in this context using the extra-notion of Bergman kernel. This idea is inspired by our new proof of Lübke's result using the asymptotic for higher tensor powers L^k of the Bergman kernel. We introduce the notion of Bergman kernel vanishing on a divisor and study its behavior when k tends to infinity. Asymptotically this Bergman kernel behaves as a characteristic function of a certain canonical set, that we call the *non-vanishing set.* The complement of this set is a certain neighborhood of the divisor whose volume is given by the Riemann-Roch formula. Finally we give a proof of the "easy" sense of the correspondence for some simple cases.

2 The case of vector bundles and the Mumford stability

For any Kähler metric g on a manifold, we let $\omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}}(z) dz_i d\bar{z}_j$ denote its corresponding Kähler form, a closed positive (1,1)-form. Now, let M

be smooth projective manifold of complex dimension n, (L, h_L) an ample hermitian line bundle on M and we denote $\omega = -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log h_L$ the curvature of h_L. Let E be a hermitian holomorphic vector bundle of rank r_E on M. We denote $N_k = \dim H^0(M, E \otimes L^k)$.

Definition 2.1. *Fix a smooth hermitian metric $h_E \in Met(E)$ on E, and define the L^2-inner product on $C^\infty(M, E \otimes L^k)$,*

$$\int_M h_E \otimes h_L^k(.,.)\frac{\omega^n}{n!}.$$

Let $(S_i)_{i=1,..,N_k}$ be an orthonormal basis of $H^0(M, E \otimes L^k)$ with respect to this L^2 inner product. We define the Bergman kernel (also called Bergman function in the litterature) of $E \otimes L^k$ as

$$B_{h_E\otimes h_L^k}(p) = \sum_{i=1}^{N_k} S_i(p)\, S_i(p)^\star \in End(E \otimes L^k)_{|p}$$

where $p \in M$. This is independent of the choice of the basis.

This can be seen as the restriction over the diagonal of $M \times M$ of

$$B_{h_E\otimes h_L^k}(p, q) = \sum_{i=1}^{N_k} S_i(p)\langle S_i(q), .\rangle_{h_E\otimes h_L^k} \in End(E \otimes L^k)$$

which is the kernel of the natural L^2-projection π_{hol} from the space of smooth sections $C^\infty(M, E\otimes L^k)$ to the space of holomorphic sections $H^0(M, E\otimes L^k)$, i.e

$$\pi_{hol}(s)(p) = \int_M B_{h_E\otimes h_L^k}(p, q)s(q)\frac{\omega^n}{n!}.$$

We note L_ω the natural contraction of $(1, 1)$ type associated to the Kähler metric, $L_\omega u = \omega \wedge u$ and $\Lambda_\omega := L_\omega^*$ the adjoint operator. When k tends to infinity, one obtains[1] the asymptotic for $B_{h_E\otimes h_L^k}$, given by

$$B_{h_E\otimes h_L^k} = k^n Id + k^{n-1}\left(\frac{1}{2}scal(\omega)Id + \Lambda_\omega F_{h_E}\right) + O(k^{n-2})$$

where $scal(\omega)$ stands for the scalar curvature of the Riemannian metric g associated to ω and F_{h_E} for the curvature of h_E. This asymptotic is actually uniform in C^∞ sense. Note that the integrals of the first two terms of the asymptotic are given by the Riemann-Roch formula. This asymptotic expansion is the key argument of our heuristic proof of the implication *existence of a Hermitian-Einstein metric ⇒ Mumford semi-stability* that we describe now.

[1] See [DLM, Ke, Wa] for the computation of the terms and [Ca] for the existence of such an asymptotic in k.

Proposition 2.1. *Let E be a holomorphic vector bundle over the projective manifold (M, L). Assume that there exists a ω-Hermitian-Einstein metric h_{HE} on E. Then E is semi-stable in the sense of Mumford.*

Proof. Let $\mathcal{F}$ be a coherent subsheaf of E of rank $0 < r_{\mathcal{F}} < r_E$. Without loss of generality we can assume that $\mathcal{F}$ is reflexive i.e torsion free and normal. We know that $\mathcal{F}$ is a subbundle of E outside a Zariski open part of M. Moreover, it is non locally free on a set S of points with codim$(S) \geq 3$. Now, from the asymptotic result described previously,

$$B_{h_{HE}\otimes h_L^k} = k^n Id + k^{n-1}\left(\frac{\mu(E)}{Vol_L(M)} + \frac{1}{2}scal(\omega)\right) Id + O(k^{n-2}) \in End(E)$$

where $\mu(E) = \frac{\mathrm{deg_L}(E)}{r_E}$ is the slope of E and $\mathrm{deg_L}(E)$ is the degree of E with respect to L. As $H^0(M, \mathcal{F}\otimes L^k) \subset H^0(M, E\otimes L^k)$, one obtains by projecting over $M\backslash S$ and the subbundle $\mathcal{F}_{|M\backslash S}$ that pointwisely for any k sufficiently large,

$$k_n Id_{\mathcal{F}} + k^{n-1}\left(\mu(E) + \frac{1}{2}scal(\omega)\right) Id_{\mathcal{F}} + Q + O(k^{n-2}) = B_{h_{HE|\mathcal{F}}\otimes h_L^k} \in End(\mathcal{F})$$

where Q is a positive auto-adjoint operator. Taking the trace, one gets directly by integration,

$$k^n Vol_L(M) r_{\mathcal{F}} + k^{n-1}\left(\mu(E) + \frac{1}{2}\int_M c_1(M)\frac{c_1(L)^{n-1}}{(n-1)!}\right) r_{\mathcal{F}} \\ \geq h^0(M, \mathcal{F}\otimes L^k) + O(k^{n-2}).$$

Now, for any k sufficiently large, the Riemann-Roch formula leads to

$$k^n Vol_L(M) r_{\mathcal{F}} + k^{n-1}\mu(E) r_{\mathcal{F}} \geq k^n Vol_L(M) r_{\mathcal{F}} + k^{n-1}\mathrm{deg}(\mathcal{F}) + O(k^{n-2})$$

and thus

$$\mu(E) \geq \frac{\mathrm{deg}(\mathcal{F})}{r_{\mathcal{F}}} = \mu(\mathcal{F}).$$

Hence E is Mumford semi-stable. □

3 The notion of Bergman kernel vanishing along a divisor

3.1 Non vanishing sets

Let (L, h_L) a hermitian ample line bundle on the Kähler manifold (M, ω) and D a smooth divisor. Let's assume that $\omega = c_1(h_L)$, i.e that ω is the curvature of (L, h_L). Let $\varepsilon(L, D)$ be the Seshadri constant of D with respect to L [De1]. By definition,

$$\varepsilon(L, D) = \sup\{c : L(-cD) \text{ is ample on the blow up } \tilde{M} \text{ of } M \text{ along } D\}.$$

By analogy with the case of Bergman kernel for subbundles, we consider the restriction over the diagonal of the integral kernel of the projection from the smooth sections of L^k vanishing at order ck on D onto the space of holomorphic sections $H^0(L^k(-ckD))$, i.e

$$\tilde{B}_h(p) = \tilde{B}_{h,k,\omega,D,M,L,c}(p) = \sum_{i=1}^{h^0(L^k(-ckD))} |S_i(p)|^2_{h_k}$$

for a point $p \in M$. Here $(S_i)_i$ is an orthonormal basis of $H^0(L^k(-ckD))$ for the inner product $\int_M h_k(.,.)dV$ with $dV = \frac{\omega^n}{n!}$ and $h_k = h_L^{\otimes k}$ is the induced metric from h_L on L^k (we see S_i as an element of $H^0(L^k)$). We denote by $||.||_{h_k}$ the L^2 norm associated to this inner product and we notice by Riemann-Roch theorem that

$$N = h^0(L^k(-ckD)) = k^n \int_{\tilde{M}} c_1(L(-cD))^n + ...$$

Remark 3.1. *Considering the operator norm of the composition of the projection $\pi : L^2(M, L^k - ckD) \to H^0(M, L^k - ckD)$ with the evaluation fiberwise ev_p, on gets that for $p \in M$,*

$$\tilde{B}_h(p) = |||ev_p \circ \pi|||^2 = \sup_{s \in H^0(L^k(-ckD))} \frac{|s(p)|^2_{h_k}}{||s||^2_{h_k}}.$$

An element realizing this extremum will be said to represent the Bergman kernel at the point p (or to be extremal at p), and is unique up to a complex constant of unit norm.

We are interested to find the asymptotic outside of D of $\tilde{B}$ when k tends to infinity.

Definition 3.1. *We define the nonvanishing set of the Bergman kernel as*

$$NV_c = \{x \in M : \frac{\tilde{B}(x)}{N} \text{ converges when } k \to \infty \text{ and its limit is non zero}\}$$

A priori this depends (of course on M) on D, c and h_L. Some natural questions arise at this stage.

Question 3.1. *Does $\frac{\tilde{B}_k}{N}$ converges almost everywhere ? When $\frac{\tilde{B}_k}{N}(x)$ converges, can we prove that this limit is 0 or 1 ? Does the Bergman kernel have a probabilistic interpretation ? What does happen on the boundary $\partial\overline{NV_c}$?*

Definition 3.2. *We set*

$$\mathcal{NV}_c^1 = \{x \in M : \text{ for all } k >> 0 \qquad \tilde{B}_h(x) = \frac{|s_x(x)|_h^2}{||s_x||_h^2}, \text{ with}$$
$$\left|\sup_p |s_x(p)|_h^2 - |s_x(x)|_h^2\right| < \epsilon(k),$$
$$\lim_{k\to\infty} \epsilon(k) = 0\}$$

Remark 3.2. *This set depends on D, c and h_L. It is closed*[2].

Definition 3.3. *We set*

$$\mathcal{NV}_c^2 = \Big\{x \in M, s.t. \qquad \exists h_D \in Met^\infty(\mathcal{O}(D)),$$
$$\omega + i\partial\bar\partial \log |s_D|_{h_D}^{2c} > 0$$
$$|s_D|_{h_D} \text{ attains its maximum at } x\Big\}$$

Remark 3.3. *This set depends on D, c and h_L. It is open because around $x \in \mathcal{NV}_c^2$, if one sets some coordinates z, $|s_D|_{h_D}^{2c} e^{-\epsilon\log(1+|z-x|^2)}$ admits its maximum on a small ball around x and still*

$$\omega + i\partial\bar\partial \left(\log\left(|s_D|_{h_D}^{2c} - \epsilon\log(1+|z-x|^2)\right)\right) > 0$$

for ϵ small enough.

Remark 3.4. *Clearly $\mathcal{NV}_c^2$ is non empty. This will show later that $\mathcal{NV}_c^1$ is not empty too*[3].

3.2 First term of the asymptotic formula for the Bergman kernel on $\mathcal{NV}_c^2$

We aim to show in this section the following result.

Theorem 1. *For all compact subset $K \subset \mathcal{NV}_c^2$, there exists $k_0 > 0$ such that for all points $p \in K$ and $k > k_0$, one can construct at p a section s_k satisfying the following properties*[4]*:*

- $s_k \in H^0(L^k - ckD)$, $||s_k||_{h_k}(p) = 1$,
- *locally at p,* $s_k(z) = \lambda_0(1 + O(|z|^2))\left(1 + O\left(\frac{1}{k^{2l}}\right)\right)\mathbf{e}^{\otimes k}$ *for any $l \geq 0$,*

[2] We can think of that space as the set of points where the representing section for the Bergman kernel at x has its maximum at x' with x' very close to x.

[3] We can think of $\mathcal{NV}_c^2$ as an open bounded part of the "anti-Kähler" cone depending on a point of M or a condition on all the positive curvatures for $L - cD$.

[4] as expected these sections do not depend on h_D. Note that for the applications we will just need $l = 1$.

- $\int_{M\setminus B(p,\log(k)/\sqrt{k})} |s_k|^2_{h_k} = O\left(\frac{1}{k^{2l}}\right)$, *and*

$$\lambda_0^{-2} = \int_{B(p,\frac{\log k}{\sqrt{k}})} e^{-kK_p(z)}dV$$

Essentially, we use Tian's idea of constructing peak sections. Remark that here the problem is not anymore local in nature because of the existence of the divisor D.

Let's fix some notations. Define $\eta \in C^2(\mathbb{R}_+,[0,1])$ a cut-off function with $\eta(r)=1$ for $0\leq r\leq r_\eta^{min}$, $\eta(r)=0$ for $r\geq 1$. The choice of r_η^{min} will be made clear during the proof. On a trivialisation around $x\in M$ we can write $h_L^k(.,.) = e^{-k\phi(x)}|.|_0$ where ϕ is psh (will be the potential of our csck metric later). We choose a point $p\in\mathcal{NV}_c^2$, call h_D the associated metric and assume that for the defining section, one has $|s_D|_{h_D}(p)=1$. Finally $B(x,r)$ will denote a geodesic ball of radius r around the point $x\in M$.

Now one can define a Kähler potential[5] $K_p(z)$ for ω which has locally the following Taylor expansion around p (Böchner holomorphic coordinates):

$$K_p(z) = |z|^2 - \frac{1}{4}R_{i\bar{j}k\bar{l}}z_i\bar{z}_jz_k\bar{z}_l + O(|z|^5)$$

Around p, consider $\mathbf{e}$ holomorphic canonical section of L with $h_L(\mathbf{e},\mathbf{e}) = e^{-K_p(z)}$.

Let's begin the proof of the theorem by considering $p\in\mathcal{NV}_c^2$ and h_D the associated metric on $\mathcal{O}(D)$, i.e for which $|s_D|_{h_D}$ has its maximum at p and value 1. Consider the smooth section

$$\sigma = \eta\left(\frac{k|z|^2}{\log(k)^2}\right)\mathbf{e}^{\otimes k} \in C^\infty(M,L^k)$$

Define the singular metrics

$$\tilde{h} := \frac{h_L}{|s_D|^{2c}_{h_D}}$$

and

$$\tilde{h}'_k := \tilde{h}^{\otimes k}e^{-\eta\left(\frac{1}{r_\eta^{min}}\frac{k|z|^2}{\log(k)^2}\right)\log\left(\frac{}{r_\eta^{min}}\frac{k|z|^2}{\log(k)^2}\right)^{(n+2)}}$$

A computation [Ti] shows that for k sufficiently large, the curvature of $\tilde{h}'_k$ is strictly positive, i.e if we set $\psi = \eta\left(\frac{1}{r_\eta^{min}}\frac{k|z|^2}{\log(k)^2}\right)\log\left(\frac{1}{r_\eta^{min}}\frac{k|z|^2}{\log(k)^2}\right)^{(n+2)}$ then $\sqrt{-1}\partial\bar{\partial}\psi \geq -\frac{Ck}{\log(k)}(\omega+\sqrt{-1}\partial\bar{\partial}\log|s_D|^{2c}_{h_D})$.

[5]In fact we just need for the following computations the first term of the Taylor expansion.

Remark 3.5. *The weight ψ is to ensure that the section we are going to build later vanishes at p, and thus is not going to destroy the peak of σ at p. In fact the term $\sqrt{-1}\partial\bar{\partial}\psi$ is going to be bounded independentely of k.*

Now, $\alpha_k = \bar{\partial}\sigma$ is a smooth (0,1)-form with value in L^k.

Lemma 3.1. *One has the estimate*[6]

$$||\alpha_k||^2_{\tilde{h}'_k} = O\left(e^{-\delta\log(k)^2}\frac{1}{k^{n-1}}\right),$$

for a certain constant $\delta > 0$.

Proof. We denote $U(p,k) = B\left(p, \frac{\log(k)}{\sqrt{k}}\right) \setminus B\left(p, r_\eta^{min}\frac{\log(k)}{\sqrt{k}}\right)$. To get an upper bound of $||\alpha_k||^2_{\tilde{h}'_k}$, one has to control

$$\begin{aligned}&\int_M \left|\bar{\partial}\eta\left(\frac{k|z|^2}{\log(k)^2}\right)\right|^2 e^{-kK_p(z)}e^{-\psi}\frac{1}{|s_D|^{2ck}_{h_D}}dV\\ \leq & cc'_\eta\int_{U(p,k)}\left|\eta'\left(\frac{k|z|^2}{\log(k)^2}\right)\right|^2\frac{k2}{\log(k)^4}|z|\frac{1}{|s_D|^{2kc}_{h_D}}e^{-kK_p(z)}dV\end{aligned}$$

since $\psi(z) = 0$ for $|z| \geq r_\eta^{min}\frac{\log(k)}{\sqrt{k}}$. Note that we have $|z| \leq \frac{\log(k)^2}{k}$ for $z \in U(p,k)$. Using the fact that $|s_D|_{h_D}$ has its maximum at p with value 1, one gets that that there exists a constant $c_{h_D} > 0$ depending on the curvature of h_D such that for all $z \in U(p,k)$,

$$\begin{aligned}|s_D|^{2c}_{h_D}(z) &\geq \left(1 - c_{(h_D,s_D)}|z|^2\right) + O(|z|^3)\\ &\geq \left(1 - c_{(h_D,s_D)}\frac{\log(k)^2}{k}\right)\left(1 + O\left(\frac{\log(k)^3}{k^{3/2}}\right)\right)\end{aligned}$$

and we notice that this constant $c_{(h_D,s_D)}$ is stricly less than 1 because $\sqrt{-1}\partial\bar{\partial}K_p(z) + \sqrt{-1}\partial\bar{\partial}\log|s_D|^{2c}_{h_D} > 0$. Thus we get for a certain constant C_1 independent of k,

$$\frac{1}{|s_D|^{2kc}_{h_D}(z)} \leq C_1 e^{c'\log(k)^2}$$

for all point $z \in B(p, \frac{\log(k)}{\sqrt{k}})$ with $1 > c' > 0$ independant of k. Hence, one just needs to evaluate

$$\begin{aligned}&e^{c'\log(k)^2}\int_{U(p,k)}\frac{k}{\log(k)^2}e^{-kK_p(z)}dV\\ \leq & e^{c'\log(k)^2}\frac{k}{\log(k)^2}\left(\frac{\log(k)^2}{k}\right)^n e^{-k\left(r_\eta^{min}\right)2\frac{\log(k)^2}{k}}\\ \leq & Ce^{(c'-\left(r_\eta^{min}\right)2)\log(k)^2}\left(\frac{\log(k)^2}{k}\right)^{n-1}\end{aligned}$$

[6]one has to keep in mind that $||\alpha_k||$ controls the defect for σ to be holomorphic.

and we can choose r_η^{min} such that $r_\eta^{min} > c'$. This ensures that we get the expected inequality. $\square$

Corollary 3.1. *For any $l \geq 0$, one has*

$$||\alpha_k||^2_{\tilde{h}'_k} = O\left(\frac{1}{k^l}\right).$$

Now, we can apply L^2-Hörmander estimates with respect to the metric $\tilde{h}'_k$. From [De1] one gets the existence of a section u_k of L^k such that

$$\begin{aligned} \bar{\partial} u_k &= \alpha_k \\ ||u_k||_{\tilde{h}'_k} &\leq \frac{C}{k}||\alpha_k||_{\tilde{h}'_k} < +\infty \end{aligned}$$

The choice of $\tilde{h}'_k$ forces u_k to vanish at p and D at order kc, and moreover from the lemma,

$$\int_M |u_k|^2_{\tilde{h}'_k} = O\left(\frac{1}{k^{n+2}}\right).$$

Consequently $|u_k| = O(|z|^2)$ on $B(p, \log k/\sqrt{k})$. Of course, we also have $||u_k||_h \leq ||u_k||_{\tilde{h}} \leq ||u_k||_{\tilde{h}'} < +\infty$[7]. Define

$$\tilde{\sigma} = \sigma - u_k,$$

which is holomorphic, vanishes on D at order kc and satisfies $|\tilde{\sigma}(p)|_{h_k} = 1$.

We know from [Ru] the following expansions when k tends to infinity:

Lemma 3.2.

$$\int_{B_{\mathbb{C}^n}(0,\log k/\sqrt{k})} |z_1^{p_1}..z_n^{p_n}|^2 e^{-k|z|^2} dz \wedge d\bar{z} = (\frac{\pi}{k})^n \frac{p_1!...p_n!}{k^{p_1+..+p_n}} + O\left(\frac{1}{k^{2p'}}\right)$$

for any $p' > p_1 + ... + p_n$.

With the two previous lemmas, we get

$$\begin{aligned} ||\tilde{\sigma}||^2_{h_k} = &\int_M \left|\eta\left(\frac{k|z|^2}{\log(k)^2}\right)\right|^2 e^{-kK_p(z)} dV \\ &- 2Re\left(\int_M \langle \eta\left(\frac{k|z|^2}{\log(k)^2}\right)\mathbf{e}^{\otimes k}, u_k\rangle_{h_L} dV\right) + ||u_k||^2_{h_k} \end{aligned}$$

Now, from last corollary, $||u_k||^2_{h_k} \leq ||u_k||^2_{\tilde{h}'_k} = O\left(\frac{1}{k^l}\right)$ for any $l \geq 0$. Moreover, by Cauchy-Schwartz

$$\begin{aligned} \left|\int_M \langle \eta\left(\frac{k|z|^2}{\log(k)^2}\right)\mathbf{e}^{\otimes k}, u_k\rangle_{h_L} dV\right| &\leq \left(\int_{B(p, \frac{\log k}{\sqrt{k}})} e^{-k|z|^2} dV\right)^{1/2} ||u_k||_{\tilde{h}'_k}\left(1 + O\left(\frac{1}{k}\right)\right) \\ &= O\left(\frac{1}{k^l}\right) \end{aligned}$$

[7]This is here where we use the fact that $|s_D|_{h_D}$ has its global maximum at p.

for any $l \geq 0$.

At the point p, we have constructed a global holomorphic section $\tilde{\sigma}$ vanishing at order kc on D and for any $l \geq 0$,[8]

$$\frac{|\tilde{\sigma}|_h^2(p)}{||\tilde{\sigma}||_h^2} = \frac{1}{\int_{B(p,\frac{\log k}{\sqrt{k}})} e^{-kK_p(z)}dV} + O\left(\frac{1}{k^l}\right) = k^n + O(k^{n-1})$$

Hence, we get that the first term of the asymptotic of is bounded from below by k^n, i.e that at $p \in \mathcal{NV}_c^2$,

$$\tilde{B}_k(p) = k^n + O(k^{n-1}).$$

3.3 Second term of the asymptotic formula for the Bergman kernel

With the same reasoning as before but using the weight

$$\psi_P = (n+2p')\eta\left(\frac{1}{r_\eta^{min}}\frac{k|z|^2}{\log(k)^2}\right)\log\left(\frac{1}{r_\eta^{min}}\frac{k|z|^2}{\log(k)^2}\right),$$

one can construct global sections $s_{k,P}$ satisfying the following properties:

- $s_{k,P} \in H^0(L^k - ckD)$, $||s_{k,P}||_h(p) = 1$,
- locally at p, $s_k(z) = \lambda_P(z_1^{p_1}..z_n^{p_n} + O(|z|^{2p'}))\mathbf{e}^{\otimes k}\left(1 + O\left(\frac{1}{k^{2p'}}\right)\right)$ for any $p' > p_1 + ... + p_n$ and the p_i are integers,
- $\int_{M\setminus B(p,\log(k)/\sqrt{k})} |s_k|^2 = O(1/k^{2p'})$ and

$$\lambda_P^{-2} = \int_{B(p,\frac{\log k}{\sqrt{k}})} |z_1^{p_1}..z_n^{p_n}|^2 e^{-kK_p(z)}dV$$

Therefore, the second term of the asymptotic can be computed exactly by following the lines[9] of Tian's paper [Ti, Lu] for a point $p \in \mathcal{NV}_c^2$, and at $p \in \mathcal{NV}_c^2$,

$$\tilde{B}_h(p) = k^n + \frac{k^{n-1}}{2}Scal(h)(p) + O(k^{n-2}).$$

Finally, we note that our construction gives a section that has at $p \in \mathcal{NV}_c^2$ the property to be close to its maximum, i.e

$$\mathcal{NV}_c^2 \subset \mathcal{NV}_c^1.$$

[8]The term in k^{n-1} appears because of the taylor expansion of $K_p(z)$ and $\det(g_{ij})$.

[9]In Tian's paper, even if it is not said, it is sufficient to use Thm 1 to get the second term because of the Riemann Roch formula, but in our case we don't know the volume of $\mathcal{NV}_c^2$ yet.

Indeed, $\frac{|\tilde{\sigma}(p)|^2_{h_k}}{||\sigma||^2_{h_k}} = k^n(1+O(k^{n-1}))$ and we know (for instance from the asymptotic on the classical Bergman kernel) that for any holomorphic section $s \in H^0(L^k)$ with $||s||_{h_k} = 1$, $\sup |s|^2_{h_k} \leq k^n + O(k^{n-1})$, so $\left| \frac{|\tilde{\sigma}(p)|^2_{h_k}}{||\sigma||^2_{h_k}} - \frac{\sup_{x\in M} |\tilde{\sigma}(x)|^2_{h_k}}{||\sigma||^2_{h_k}} \right| = O(1/k)$.

Note that for a point p in $\mathcal{NV}^1_c$ and a sequence of peakes sections s_k at p constructed as before, if the sequence $|s_k|^{2/k}_{h_k}$ converges to a smooth limit which is positive on $M \setminus D$, then it gives a smooth metric $\left(\frac{|s_k|^2_{h_k}}{|s_D|^{2kc}_0}\right)^{1/k} \langle , \rangle_0$ on $\mathcal{O}(D)$ (for which the norm of s_D takes its maximum at x) and since the log of the norm of a holomorphic section is psh, p belongs to $\mathcal{NV}^2_{c,h_D}$[10].

Hence, we have seen that on compact subsets of $\mathcal{NV}^2_c$, we can get by the procedure developped in [Ti, Ru] an asymptotic expansion of $\tilde{B}_k$ in the C^∞ topology.

4 The 0-1 law for the Bergman function $\frac{\tilde{B}_k}{N}$

4.1 A uniqueness result for peaked sections

We aim to show that if one has a section $S \in H^0(M, L^k - ckD)$ with a "peak" at a point p, and with L^2 norm 1, then the L^2 norm of S is concentrated around p. This is completely elementary.

Lemma 4.1. *Suppose $s_k \in H^0(M, L^k - ckD)$ is the peak section at $p \in \mathcal{NV}^2_c$ constructed as above in Theorem 1. Let s_0 be another section of L^k such that s_0 vanishes at p. Then*

$$\int_M \langle s_k, s_0 \rangle_{h_k} = O\left(\frac{1}{k}\right) ||s_0||_{h_k}$$

Proof. See [Ru]. □

Suppose that $|S(p)|^2_{h_k} = k^n + O(k^{n-1})$. It is clear from Lemma 3.2 that

$$\int_M \langle s_k - S, S \rangle_{h_k} = \int_M O(k^{n-1}) O(|z|^2) e^{-k|z|^2} dV = O(1/k).$$

Using previous lemma with $s_0 = s_k - S$ one gets

Proposition 4.1. *Assume that $S \in H^0(M, L^k - ckD)$ with $||S||_{h_k} = 1$ satisfies $|S(p)|^2_{h_k} = k^n + O(k^{n-1})$ for $p \in \mathcal{NV}^2_c$. Then*

$$||S - s_k||^2_{h_k} = O(1/k)$$

for s_k the peaked section constructed at p as before.

[10] One expects at this stage $\overline{\mathcal{NV}^2_c} = \mathcal{NV}^1_c$ as we shall prove later.

Since we know that at each point of the non-vanishing set, we can construct a peak section, we obtain:

Corollary 4.1. *Let p be a point in $\mathcal{NV}_c^2$. Then the representing section $S_{c',p}$ at p for $\tilde{B}_{k,c'}$ converges in L^2 norm to the representing section $S_{c,p}$.*

4.2 Some natural inclusions

Since $\sup_M |S|^2_{h_k} \geq 1/V$ for a section $S \in H^0(L^k)$ with L^2 norm equal to 1, one has directly

$$\mathcal{NV}_c^1 \subset NV_c$$

and from last section we know $\mathcal{NV}_c^2 \subset \mathcal{NV}_c^1$. Also, it is clear that

$$\mathcal{NV}_0^1 = \mathcal{NV}_0^2 = NV_0 = M$$

and

$$\mathcal{NV}^1_{\varepsilon(L,D)} = \mathcal{NV}^2_{\varepsilon(L,D)} = \emptyset.$$

From another part, it is clear that for $c' > c$,

$$\mathcal{NV}_{c'}^2 \subset \mathcal{NV}_c^2$$

Moreover, if $\omega + i\partial\bar{\partial}\log|s_D|^{2c}_{h_D} > 0$ then we still have $\omega + i\partial\bar{\partial}\log|s_D|^{2c'}_{h_D} > 0$ for $c < c' < c + \epsilon$ for ϵ small enough one gets that

$$\cup_{c'>c}\mathcal{NV}_{c'}^2 = \mathcal{NV}_c^2.$$

Proposition 4.2. *One has $\cap_{c'<c}\mathcal{NV}_{c'}^2 = \overline{\mathcal{NV}_c^2}$.*

Proof. The only difficult part is to show that $\overline{\mathcal{NV}_c^2} \subset \cap_{c'<c}\mathcal{NV}_{c'}^2$. Suppose that there exists a metric $h_\infty \in Met(\mathcal{O}(D))$ non necessarily smooth such that $\omega + i\partial\bar{\partial}\log|s_D|^{2c}_{h_\infty} \geq 0$ with $|s_D|_{h_\infty}$ has its maximum at x. Then, since $\omega > 0$, for $c' < c$ one gets directly

$$\omega + i\partial\bar{\partial}\log|s_D|^{2c'}_{h_\infty} = i\partial\bar{\partial}\left(\phi_L + \log|s_D|^{2c'}_{h_\infty}\right) > 0.$$

Now, using [De1] one can approximate locally (i.e we use a finite covering Ω_i of M by pseudoconvex open sets) the psh function $\phi_L + \log|s_D|^{2c'}_{h_\infty}$ using a sequence of psh function $\phi_{m,i} = \frac{1}{2m}\log\sum_j|\sigma_j|^2$ for $(\sigma_j)_j$ a Hilbert basis of sections of L^k in $L^2_{\Omega_i}\left(m\phi_L + m\log|s_D|^{2c'}_{h_\infty}\right)$. Note that on compact subsets of Ω_i, the boundness from above of $\phi_L + \log|s_D|^{2c'}_{h_\infty}$ implies the uniform convergence of $\sum|\sigma_j|^2$ on Ω_i[11]. Finally, since ϕ_L is smooth, one gets that $\phi_{m,i} - \phi_L$ converges uniformly and thus has its maximum at x. The pointwise convergence on the whole manifold of the $\phi_{m,i}$ implies that this maximum is global. □

[11]By mean value inequality, the L^2 topology is here stronger than topology of uniform convergence on compact subsets.

Corollary 4.2. *If $c' < c$, then*

$$\overline{\mathcal{NV}_c^2} \subset \mathcal{NV}_{c'}^2.$$

Corollary 4.3. *One has*

$$\mathcal{NV}_c^1 = \overline{\mathcal{NV}_c^2}.$$

4.3 Behavior of the Bergman function $\frac{\tilde{B}_k}{N}$

Fix $\varepsilon(L, D) > c > 0$. We know that for all $p_0 \in M$ and k sufficiently large, $\frac{\tilde{B}_{k,c,D}}{N}(p_0) \in [0, 1]$. Suppose that $\frac{\tilde{B}_{k,c,D}}{N}(p)$ does admit a subsequence converging to a constant $\delta > 0$ for a point $p \in M \setminus \overline{\mathcal{NV}_c^2}$. We will show that we obtain a contradiction by proving that we can construct a peaked section at p and thus p must belong to $\overline{\mathcal{NV}_c^2}$.

Indeed, for this subsequence $\gamma(k) \in \mathbb{N}$, the representing sections $s_{\gamma(k),p}$ at p are such that $|s_{\gamma(k),p}|_{h_{\gamma(k)}}$ attain its maximum at $p_{\gamma(k)} \in \overline{\mathcal{NV}_c^2}$.[12]

From Corollary 4.2, there exists $c' < c$ sufficiently close to c, such that

$$p_{\gamma(k)} \in \mathcal{NV}_{c'}^2 \text{ and } p \in M \setminus \overline{\mathcal{NV}_{c'}^2}, \tag{1}$$

$$\exists \text{ a subsequence } \gamma'(k) \text{ of} \gamma(k), \text{ s.t } \lim_{k \to \infty} p_{\gamma'(k)} = p_\infty \in \mathcal{NV}_{c'}^2. \tag{2}$$

The sections $s_{\gamma'(k),p}$ are also vanishing at order $\gamma'(k)c'$ and –up to considering a subsequence– we can assume that $|s_{\gamma'(k),p}|_{h_{\gamma'(k)}}(p_{\gamma'(k)}) = \delta' \gamma'(k)^n (1 + O(1/k)) \geq \delta \gamma'(k)^n$. It means that $s_{\gamma'(k),p}$ has another peak at $p_{\gamma'(k)}$. From another hand, there exists a peaked section $s_{\gamma'(k),p_{\gamma'(k)}}$ at $p_{\gamma'(k)}$ such that

$$S = s_{\gamma'(k),p} - \delta' s_{\gamma'(k),p_{\gamma'(k)}}$$

vanishes at $p_{\gamma'(k)}$ and has pointwise norm $\delta\gamma'(k)^n$ at p. Indeed, we construct this section $s_{\gamma'(k),p_{\gamma'(k)}}$ as in the first paragraph but with the weight

$$\psi_1 = \eta \left(\frac{1}{r_\eta^{min}} \frac{k|z - p_{\gamma'(k)}|^2}{\log(k)^2} \right) \log \left(\frac{1}{r_\eta^{min}} \frac{k|z - p_{\gamma'(k)}|^2}{\log(k)^2} \right)^{(n+2)}$$
$$\times \eta_1 \left(2 \frac{k|z - p|^2}{\log(k)^2} \right) \log \left(2 \frac{k|z - p|^2}{\log(k)^2} \right)^{(n+2)}$$

[12] There is a subtlety here since $|s_{\gamma(k),p}|_{h_{\gamma(k)}}$ could vanish outside D. But we can add a cut-off function $0 \leq \tilde{\eta} \leq O(1/k)$ such that $\tilde{\eta}$ is non zero where $|s_{\gamma(k),p}|_{h_{\gamma(k)}}$ vanishes on $M \setminus D$ and $\tilde{\eta}$ vanishes on D and around $p_{\gamma(k)}$. If we choose $\tilde{\eta}$ carefully, i.e bound its derivatives, we can assume that $\omega + \sqrt{-1}\partial\bar{\partial} \log(|s_{\gamma(k),p}|^2_{h_{\gamma(k)}} + \tilde{\eta}) > 0$ and by construction $\frac{|s_{\gamma(k),p}|^2_{h_{\gamma(k)}} + \tilde{\eta}}{|s_D|_0^{2kc}} \langle , \rangle_0$ gives a well defined metric on $\mathcal{O}(D)$. Finally, the function $|s_{\gamma(k),p}|^2_{h_{\gamma(k)}} + \tilde{\eta}$ has still its maximum at $p_{\gamma(k)}$, and thus, by definition, $p_{\gamma(k)} \in \mathcal{NV}_c^2$

where $\eta_1 \in C^2(\mathbb{R}_+, [0,1])$ is a cut-off function with $\eta_1(r) = 0$ for $r \leq 1/2$ or $r \geq 1$. This weight will force the constructed section to vanish also at p. Note that this is possible since we have the convergence of $p_{\gamma'(k)}$ in $\mathcal{NV}^2_{c'}$.

From Proposition 4.1, the section S satisfies

$$||S||_{h_{\gamma'(k)}} < ||s_{\gamma'(k),p}||_{h_{\gamma'(k)}} - \frac{\delta'}{2} + O\left(\frac{1}{\gamma'(k)}\right)$$

Hence, there exists a constant $\lambda > 1$ such that $||\lambda S||_{h_{\gamma'(k)}} = 1$ and also $|\lambda S(p)|_{h_{\gamma'(k)}} > \delta\gamma'(k)^n$. Of course, we can assume that $|S(p)|_{h_{\gamma'(k)}}$ is the maximum of the function $|S|_{h_{\gamma'(k)}}$ on $M \setminus \overline{\mathcal{NV}^{c'}_2}$[13], and even on M if we do the same reasoning for the (finite number[14] of) points where this function has a local maximum on $\overline{\mathcal{NV}^2_{c'}}$ bigger than $\delta\gamma'(k)^n$. Hence, by definition, p belongs to $\mathcal{NV}^1_{c'} = \overline{\mathcal{NV}^2_{c'}}$ and we get a contradiction with condition (1).

Finally, we have proved, using the previous result of the asymptotic of the Bergman kernel on the non vanishing set, that

Theorem 2 (0–1 law)**.** *If* $p \in \mathcal{NV}^2_c$*, then* $\lim_{k\to} \frac{\tilde{B}_k}{N}(p) = 1$*. If* $p \in M \setminus \overline{\mathcal{NV}^2_c}$*, then* $\lim_{k\to} \frac{\tilde{B}_k}{N}(p) = 0$.

By integration of $\frac{\tilde{B}_k}{N}$ and using Riemann-Roch formula, we know that

$$Vol(\mathcal{NV}^2_c) + \lim_{k\to\infty} \int_{\partial\mathcal{NV}^2_c} \frac{\tilde{B}_k}{N} = Vol(L - cD)$$

which leads to

$$Vol(\mathcal{NV}^2_c) + Vol(\partial\overline{\mathcal{NV}^2_c}) \geq Vol(L - cD) \geq Vol(\mathcal{NV}^2_c)$$

Now, using Proposition 4.2, we know that

$$Vol(\overline{\mathcal{NV}^2_c}) \leq Vol(\mathcal{NV}^2_{c'}) \leq Vol(L - c'D)$$

for all $c' < c$. The function $c' \mapsto Vol(L - c'D)$ is continuous, so we get that

$$Vol(\overline{\mathcal{NV}^2_c}) \leq Vol(L - cD)$$

and consequently

$$Vol(\overline{\mathcal{NV}^2_c}) = Vol(L - cD).$$

Now, from Corollary 4.2, $Vol(\mathcal{NV}^2_c) \geq Vol(\overline{\mathcal{NV}^2_{c'}}) \geq Vol(L - c'D)$ for all $c' > c$ and by continuity, $Vol(\mathcal{NV}^2_c) \geq Vol(L - cD)$.

Finally, this gives

[13] just because we can assume that p is the point where the original representing section $s_{\gamma(k),p}$ has its maximum on $M \setminus \overline{\mathcal{NV}^2_{c'}}$

[14] for each point, we substract $\delta/2$ from the L^2 norm.

Corollary 4.4. *The boundary of* $\overline{\mathcal{NV}_c^2}$ *is Lebesgue negligible. The volume of* $\mathcal{NV}_c^2$ *with respect to* ω *is the algebro-geometric quantity* $Vol(L - cD)$.

As we mentioned previously, note that $\mathcal{NV}_c^2$ depends clearly on h_L. This leads to

$$\mathcal{NV}_c^2(h_L) = \mathcal{NV}_{c/r}^2(h_L^{\otimes r})$$

and thus

Corollary 4.5. *For any* $z \in M \setminus D$ *and* $0 \leq c < \varepsilon(L, D)$, *there exists a metric* h_L *on* L *such that* $\lim_{k \to} \frac{\tilde{B}_{h_L,c}(z)}{k^n} = 1$ *or equivalently,* $x \in \mathcal{NV}_c^2(h_L)$.

Remark 4.1. *Some information for the full Bergman kernel* $\tilde{B}_k(x, y)$ *on* $M \times M$ *can be deduced from our work.*

4.4 The Bergman exhaustion function

Using the non-vanishing set, we introduce now a function that measures the distance of a point of M to the divisor.

Definition 4.1. *Define for a point* $p \in M$

$$\rho_D(p) = \sup_{c \geq 0} \{p \in \overline{\mathcal{NV}_c^2}\}$$

Note that this function is also dependent on ω.

Proposition 4.3. *The function* $p \to Exh_D(p)$ *is a continuous function.*

Proof. We note that $\rho_D(p) \leq c$ is equivalent to $p \in \cap_{c' < c} \mathcal{NV}_{c'}^2$ which is closed from Proposition 4.2. Now, if $\rho_D(p) > c$, there exists $c' > c$ such that $p \in \mathcal{NV}_{c'}^2$ and thus $p \in \cup_{c' > c} \mathcal{NV}_{c'}^2$. If $p \in \cup_{c' > c} \mathcal{NV}_{c'}^2$, then $\rho_D(p) > c$. Hence, $\rho_D(p) > c$ is equivalent to $p \in \cup_{c' > c} \mathcal{NV}_{c'}^2$ which is open. □

Lemma 4.2. *We have*

$$\rho_D(p) = \sup_{c \geq 0} \limsup_{k \to \infty} \frac{c\tilde{B}_{h_L,k.c}(p)}{k^n} = \limsup_{k \to \infty} \sup_{c \geq 0} \frac{c\tilde{B}_{h_L,k.c}(p)}{k^n}.$$

Proof. The first equality is clear from Theorem 2 and the fact that $\frac{c\tilde{B}_{h_L,k.c}(p)}{k^n}$ is bounded in c and k. The second equality is also a consequence a Theorem 2. □

Proposition 4.4. *Let* $p \in M \setminus D$ *and* $0 < c < \varepsilon(L, D)$. *Assume that for a fixed* $k_0 \geq 1$,

$$\tilde{B}_{h_L,k_0.c}(p) \geq \kappa.$$

Then $p \in \mathcal{NV}_{c\kappa/k_0^n}^2$.

Sketch of the proof. One aims to show that $\rho_D(p) \geq c\kappa$. By assumption, note that $k_0 c \geq 1$ and wlog $c > \rho_D(p)$. With Lemma 4.2, it turns out that it is sufficient to prove that if $c_{max(k_0)} < c$ is the maximum of c' such that $k_0 c' \in \mathbb{N}^*$ and $c' < \rho_D(p)$, then

$$c_{max(k_0)} \frac{\tilde{B}_{c_{max(k_0)},k_0}(p)}{k_0^n} \geq \left(c_{max(k_0)} + \frac{q}{k_0}\right) \frac{\tilde{B}_{c_{max(k_0)+\frac{q}{k_0}},k_0}(p)}{k_0^n}$$

for any integer $1 \leq q \leq [k_0(\varepsilon(L, D) - c_{max(k_0)})]$[15]. Therefore, it is sufficient to prove that

$$c_{max(k_0)} \left(\tilde{B}_{c_{max(k_0)},k_0}(p) - \tilde{B}_{c_{max(k_0)+1/k_0},k_0}(p)\right) \geq \frac{q}{k_0} \tilde{B}_{c_{max(k_0)+\frac{1}{k_0}},k_0}(p).$$

Now, for at p, we know that we can build a peak section vanishing at order $c_{max(k_0)}k_0$ on D since $c_{max(k_0)} < \rho_D(p)$ and because of the definition of c_{max}, we know that this section vanishes exactly at order $c_{max(k_0)}k_0$ and not more. Hence,

$$\left(\tilde{B}_{c_{max(k_0)},k_0}(p) - \tilde{B}_{c_{max(k_0)+1/k_0},k_0}(p)\right) = k_0^n(1 + \delta_p/k_0).$$

From the result of Catlin [Ca], we know[16] that δ_p is going to be bounded (from below) on M, say by the constant δ. On the other hand, we need to study the behavior of $\tilde{B}_{c_{max(k_0)+\frac{1}{k_0}},k_0}(p)$ when k_0 is not too large. Let's call $\Gamma(c) = \{h_D \in Met^\infty(\mathcal{O}(D)), \omega + i\partial\bar{\partial}\log|s_D|^{2c}_{h_D} > 0\}$. At p, we know that

$$\sup_{h_D \in \Gamma(c_{max(k_0)}+q/k_0)} |s_D(p)|_{h_D}^{2c_{max(k_0)}+\frac{2q}{k_0}} < 1$$

for $\max_M |s_D|_{h_D} = 1$ and we call $\gamma(p, k_0, q)$ this value. Hence Hörmander's estimates gives us a "non-peak" section $s \in H^0(L^{k_0} - (c_{max(k_0)}k_0 + q)D)$ such that $|s(p)|^2 \leq k_0^n \gamma(p, k_0, q)^{k_0}$ and we can assume wlog that all the other sections of the basis vanish at p. In fact we expect a uniform exponential decrease, i.e

Claim. $\gamma(p, k_0, q)^{k_0} \leq \left(1 - \frac{A+q}{k_0}\right)^{k_0}$ with $A \geq 0$ independent of p and $k_0 > k_0'$ where k_0' is independant of p.

Let's assume the claim proved. Then

$$\frac{q}{k_0}\tilde{B}_{c_{max(k_0)+\frac{1}{k_0}},k_0}(p) \leq \frac{q}{k_0}\left(1 - \frac{A+q}{k_0}\right)^{k_0} k_0^n \leq \frac{q}{e^q} e^{-A} k_0^{n-1}$$

[15]The result is clear when k_0 tends to infinity, i.e the function $c\mathbf{1}_{\mathcal{N}\mathcal{V}^2_{\rho_D(p)}}$ is decreasing for $c > \rho_D(p)$. In fact we expect an exponential decrease of the Bergman kernel at a finite k_0 for $c > \rho_D(p)$

[16]The terms of the asymptotic are continuous functions

Now for k_0 sufficiently large (and this can be done indepentely of p) we get

$$c_{max(k_0)} k_0^n (1+\delta/k_0) \geq \frac{q}{e^q} e^{-A} k_0^{n-1}$$

for any $1 \leq q \leq [k_0(\varepsilon(L,D) - c_{max(k_0)})]$.

Corollary 4.6. *There exists $k_1 \in \mathbb{N}$ depending on (L, M, D, h_L) such that for all $p \in M$,*

$$\rho_D(p) = \sup_{k \geq k_1} \sup_{c \geq 0} \frac{c\tilde{B}_{h_L,k.c}(p)}{k^n}.$$

The function $Exh_{\mathcal{NV}_c^2}(p) = -\log(\rho_D(p) - c)$ defined on $\mathcal{NV}_c^2$ is a continuous exhaustion function.

4.5 Relation with Lelong numbers

For a Kähler form ω, we consider the space of strictly ω-plurisubharmonic functions

$$Ka_{[\omega]} = \{\phi \in L^1(M) : \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\},$$

Recall the Lelong number for a psh function ϕ at a point x_0,

$$\nu(\phi, x) = \liminf_{x \to x_0} \frac{\log \phi(x)}{\log |x - x_0|} = \lim_{r \to 0+} \frac{\sup_{B(x_0,r)} \phi(x)}{\log r}$$

and define $\nu(\phi, D) = \inf_{x \in D} \nu(\phi, x)$. Then, one can consider the canonical *equilibrium metric with poles on D of order c* (see [Berm, Section 4.1]) given by

$$\phi_{equil,D,c}(x) = \sup_{\psi \in Ka_{[\omega]}} \{\psi(x) : \quad \nu(\psi, D) \geq c, \quad \psi \leq -\log h_L\}$$

Then it is straightforward to check the equalities

$$\begin{aligned}\mathcal{NV}_c^2 &= \{x \in M : \exists \psi \in Ka_{[\omega]}, \nu(\psi, D) \geq c, \text{ and} \sup_M \psi = \psi(x)\} \\ &= \{\phi_{equil,D,c} = -\log h_L\}\end{aligned}$$

5 Some examples

5.1 The case of $\mathbb{P}^1$

Let's consider the elementary case of $\mathbb{P}^1$ without a point. Choose for ϕ the potential of the Fubini-Study metric. Then a 'limit' -and naive- choice for the metric on h_D leads to consider

$$\frac{|z|^{2c}}{1 + |z|^2}$$

which has its maximum on the circle of radius $\sqrt{\frac{c}{1-c}}$. Hence one can prove that $\mathcal{NV}_c^2 = \{z : |z|^2 > c/(1-c)\}$. Note that in that case, we have an explicit formula. At the point z_0, the defining section for the Bergman kernel is

$$s_{z_0}(z) = \sum_{i=kc}^{k} C_k^i \frac{z^i}{z_0^{k-i}}$$

and hence

$$\tilde{B}_{\mathbb{P}^1,h_{FS},k,c}(z) = \frac{|\sum_{i=kc}^k C_k^i \frac{1}{z^{k-2i}}|^2}{(1+|z|^2)^k \sum_{i=kc}^k C_k^i \frac{1}{z^{2k-2i}}}.$$

We have computed the following expansion for $\mathbb{P}^1$ without 1 point (using the coordinates $x = \frac{|z|}{\sqrt{1+|z|^2}}$).

Proposition 5.1.

$$\tilde{B}_{k,c}(z) - \frac{1}{2}\left(\tilde{B}_{k,c+\frac{1}{k}} - \tilde{B}_{k,c}\right)(z) = k\mathbf{1}_{x>c} + \mathbf{1}_{x>c} + \left(c - \frac{1}{2}\right)\delta(x-c) + \frac{c-c^2}{2}\delta'(x-c) + O(1/k)$$

where δ is the Dirac function and δ' its derivative with respect to x.

Note that no higher order derivatives of the Dirac function appear. If we call

$$\varepsilon_c(x) = \left(c - \frac{1}{2}\right)\delta(x-c) + \frac{c-c^2}{2}\delta'(x-c)$$

then we check that for $0 \le \mathbf{c}_0 \le \varepsilon(L, D)$,

$$\int_0^{\mathbf{c}_0} \varepsilon_c(x)dc = 0$$

if $x \le \mathbf{c}_0$ or $x > \mathbf{c}_0$. This implies the slope-semistability inequality in that case.

Remark 5.1. *In that setting, we also get*

$$Exh_{0,c}(z) = -\log(x-c).$$

For $\mathbb{P}^1$ without 2 points, say a, b, one can remark that

$$\tilde{B}_{2k,\mathbb{P}^1\setminus\{a,b\}} \le \tilde{B}_{k,\mathbb{P}^1\setminus\{a\}}\tilde{B}_{k,\mathbb{P}^1\setminus\{b\}}$$

So for c small, $\mathcal{NV}_c^2(a) \cup \mathcal{NV}_c^2(b) \subset \mathcal{NV}_c^2(a,b)$ and considering the volume, this leads to $\mathcal{NV}_c^2(a,b) = \mathcal{NV}_c^2(a) \cup \mathcal{NV}_c^2(b)$.

6 Other remarks

6.1 Another point of view with singular Bergman kernels

For the singular metric $\tilde{h} := h_L/|s_D|^{2c}_{h_D}$, we can consider the Bergman kernel $B_{sing}(\tilde{h}) = \sum |s_i|^2_{\tilde{h}}$ where s_i are L^2 orthonormal with respect to $\tilde{h}$ and dV. It is similar to the usual Bergman kernel but for the singular metric $\tilde{h}$.

For any metric $h_D \in Met(\mathcal{O}(D))$, if one assumes $|s_D| \leq 1$, one gets clearly

$$\tilde{B}(h) \geq |s_D|^{2kc}_{h_D} B_{sing}(h/|s_D|^{2c}_{h_D})$$

Now, if $p \in \mathcal{NV}^2_c$ and h_D is the associated metric at that point, then we have

$$\tilde{B}(h)(p) \geq B_{sing}(h/|s_D|^{2c}_{h_D})(p)$$

Now, for a local trivialisation around the point p, we can choose local holomorphic coordinates z s.t. $z(p) = 0$ and such that the metric around p is euclidean wrt z at $z = 0$. For the potential $\tilde{\phi}$ of the $\tilde{h}$ around p, we have $\tilde{\phi} = \phi_0 + o(|z|^2)$ where ϕ_0 is a quadratic form with associated eigenvalues $\lambda_1, ..., \lambda_n$ (wrt ω). For a section $s \in H^0(L^k)$ given by a holomorphic function f_0, one has since $|f_0|^2$ is psh,

$$|f_0(0)|^2 \leq \frac{\int_{|z|<\log(k)/\sqrt{k}} |f_0(z)|^2 e^{-k\phi_0}}{\int_{|z|<\log(k)/\sqrt{k}} e^{-k\phi_0}}$$

The numerator of the RHS can be estimated from above by $(1+\epsilon_k)\int_M |s|^2_{h_k} dV$ as k tends to infinity with $\lim_{k\to\infty} \epsilon_k = 0$. Now by assumption on p, all eigenvalues λ_i are positive so we have an estimate for the denominator of the RHS of last equation, which is up to a multiplicative constant $\frac{1}{k^n \lambda_1 ... \lambda_n} + O(k^{-n-1})$. On the other hand, as it is explained in [Bern], one can build peak sections of L^k as does Tian in [Ti] for the singular metric $\tilde{h} \in Met(L)$ which has positive curvature. Hence, the singular Bergman kernel is going to converge at p towards $k^n \lambda_1(p)...\lambda_n(p) = k^n$ at the first order[17]. On the whole manifold the singular Bergman kernel converges pointwisely towards the absolute continuous part of the current $i\partial\bar{\partial}\tilde{\phi}$.

6.2 About extending the sections

When one tries to apply L^2-Hörmander estimates in our setting, it appears that we have two natural Hilbert spaces, the space of L^2 sections with respect to h and the space of L^2 sections wrt the singular metric $\tilde{h}$. If we consider f a section vanishing of L^k vanishing at order a kc on D, then we can find u_1 and u_2 such that $\bar{\partial} u_1 = \bar{\partial} u_2 = f$ and for the L^2 norms,

$$\begin{aligned} ||u_1||^2_h &< c_1(k)||f||^2_h \\ ||u_2||^2_h \leq ||u_2||^2_{\tilde{h}} &< c_2(k)||f||^2_{\tilde{h}} \end{aligned}$$

[17] One gets only the first term with this method.

which implies that there exists a constant δ_k with $||u_2 + \delta_k||^2_h < c_1||f||^2_h$ and u_2 vanish at order kc on D. If c_k is small enough with respect to $1/k^n$, one could build a peak section from u_2 and control completely its asymptotic.

Question 6.1. *Is the constant δ given by algebraic geometry ?*

The Ohsawa-Takegoshi-Manivel theorem [De1] applied at a point $p \in \mathcal{NV}^2_c$ and the singular metric $\tilde{h} = \frac{h_k}{|s_D|^{2kc}}$ leads directly to the existence of a global holomorphic section S of L^k satisfying

$$\int_M |S|^2_{\tilde{h}} k^n \omega^n \leq C(M,n)|S(p)|^2_{\tilde{h}}$$

i.e S vanishes at order ck on D and since $|s_D|_{h_D}(p) = 1$,

$$\tilde{B}(p) \geq \frac{k^n}{C(M,n)}.$$

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