About Bergman geodesics and homogeneous complex Monge-Ampère equations

[A survey following Phong-Sturm and Berndtsson]

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Abstract

The aim of this survey is to review the results of Phong-Sturm and Berndtsson on the convergence of Bergman geodesics towards geodesic segments in the space of positively curved metrics on an ample line bundle. As previously shown by Mabuchi, Semmes and Donaldson the latter geodesics may be described as solutions to the Dirichlet problem for a homogeneous complex Monge-Ampère equation. We emphasize in particular the relation between the convergence of the Bergman geodesics and semi-classical asymptotics for Berezin-Toeplitz quantization. Some extension to Wess-Zumino-Witten type equations are also briefly discussed.

Introduction

Let $L \to X$ be an ample line bundle on a smooth projective manifold X of complex dimension n and denote by $\mathcal{H}_\infty$ the space of all (smooth) Hermitian metrics h on L with positive curvature form. Fixing a reference metric h_0 with curvature form ω_0, any other metric may be written as $h_\phi = e^{-\phi} h_0$ with curvature form $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$, using the convention which makes the curvature form a *real* $2-$form. Hence, it will be convenient to make the identification

$$\mathcal{H}_\infty = \{\phi \in C^\infty(X) : \omega_\phi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

realizing $\mathcal{H}_\infty$ as a subspace of the space of all, say continuous, ω_0-psh functions ϕ on X, i.e. $\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi \geq 0$ in the sense of currents. The space $\mathcal{H}_\infty$ can be equipped with a Riemannian metric (Cf. the work of T. Mabuchi, S. Semmes and S.K. Donaldson): for any tangent vector at ϕ, which may be identified with a smooth function ψ on X

$$\|\psi\|_\phi^2 = \int_X |\psi|^2 \omega_\phi^n$$

Hence one can speak, at least at the formal level, of geodesics in $\mathcal{H}_\infty$. Moreover, the geodesic curvature $c(t) = c(\phi_t)$ is given by the expression

$$c(\phi_t) = \ddot{\phi}_t - |\bar{\partial}_x \dot{\phi}_t|^2_{\omega_\phi} \tag{1}$$

which hence vanishes precisely when ϕ_t is a geodesic. By complexifying the variable t, a geodesic ϕ_t connecting two elements in $\mathcal{H}_\infty$ may be obtained by solving the Dirichlet problem for a complex Monge-Ampère operator in the $(x, t)-$variables (see section 1.1).

The previous setup extends immediately to the "transcendental" setting where ω_0 is any fixed Kähler form on a compact complex manifold X, i.e. without assuming that $2\pi\omega_0$ is an *integral* class. Then $\mathcal{H}_\infty$ is by definition the open convex space in the cohomology class $[\omega_0]$ consisting of all ω_0-psh functions.

On the other hand, when $2\pi\omega_0$ be an *integer* class, i.e. ω_0 is the curvature of a metric h_0 on an ample line bundle $L \to X$, the setup may be "quantized" as follows. For any given positive integer k, the infinite dimensional space $\mathcal{H}_\infty$ is replaced by a certain *finite-dimensional* symmetric space $\mathcal{H}_k$: the space of *Bergman metrics at level k*. By definition, these metrics are pull-backs of Fubini-Study type metrics by the embeddings of the manifold into the projective space $\mathbb{P}H^0(L^k)^\vee$ (Cf. section 1.3). A result of T.Bouche and G. Tian asserts that any element ϕ in $\mathcal{H}_\infty$ can be seen as a canonical limit of a sequence of elements $P_k(\phi)$ in the spaces $\mathcal{H}_k$. The symmetric spaces $\mathcal{H}_k$ come equipped with an intrinsic Riemannian structure. In particular, any two elements in $\mathcal{H}_k$ may be connected by a unique *Bergman* geodesic (at level k). Hence, it is natural to ask if the whole Bergman geodesic ψ_t connecting $P_k(\phi_0)$ and $P_k(\phi_1)$ in $\mathcal{H}_k$ converges to the geodesic ϕ_t in $\mathcal{H}_\infty$ when k tends to infinity (and not only its end points)? The question was answered in the affirmative by Phong-Sturm [PS06]. Their result was subsequently refined by Berndtsson [Ber09b, Ber09c] (see also the very recent work [Ber09c]) using a completely different approach.

The aim of the present notes is to survey these two latter results on convergence of segments of Bergman geodesics. There is also a stronger convergence result in the setting of toric varieties due to J. Song and S. Zelditch that will not be discussed here. For a general introduction to the circle of ideas in Kähler geometry surrounding all these results see the recent survey [PS08]. Let us just briefly mention that an important feature of Kähler geometry is that several important functionals (Lagrangians) on $\mathcal{H}_\infty$, whose critical points yield canonical Kähler metrics, turn out to be convex along geodesics in $\mathcal{H}_\infty$ and $\mathcal{H}_k$. In particular, the use of geodesic *segments* in $\mathcal{H}_k$ is underlying in Donaldson's seminal work [Don01] to prove *uniqueness* (up to automorphisms) of constant scalar curvature metrics in $\mathcal{H}_\infty$ (essentially by connecting any given two such metrics by a geodesic segment). On the other hand geodesic *rays* in $\mathcal{H}_k$ are closely related to the still unsolved existence problem for canonical Kähler metrics, the so-called Yau-Tian-Donaldson conjecture. The point is to study "properness" i.e. the growth of energy functional along the geodesic rays which turns out to be related to algebro-geometric notions of stability (notably asymptotic Chow-Mumford stability as conjectured by Yau [Yau87] long time ago).

The organization of the survey is as follows. We begin by briefly recalling the "quantum formalism" as it offers a suggestive description of the results to be discussed. Then the key steps in the proofs of first Phong-Sturm and then Berndtsson's results are indicated essentially following arguments in the original papers. In section 2.2 it is explained how to deduce a slightly weaker version of Berndtsson's convergence result using asymptotic formulas for products of Toeplitz operators. These formulas are well-known in the context of Berezin-Toeplitz quantization of Kähler manifolds. This latter approach is actually analytically far more involved than Berndtsson elegant curvature estimate, but hopefully it may shed some new light on Berndtsson's convergence result and its relation to quantization. Some extension to Wess-Zumino-Witten type equations are also briefly discussed in the last section.

The "quantum formalism"

The state space of a *classical* physical system is mathematically described by a symplectic manifold X equipped with a symplectic form ω. An "observable" on the state space (X, ω) is just a real-valued function on X. From this point of view *quantization* is the art of associating a Hilbert space $H(X, \omega)$ (the "quantum state space") to (X, ω) and Hermitian operators on $H(X, \omega)$ to real-valued function on X. Moreover, the quantizations should come in families paremetrized by a small parameter h ("Planck's constant") and in the limit $h \to 0$ the classical setting should emerge from the quantum one, in a suitable sense (the "correspondence principle"). One possibility to make this latter principle more precise is to demand that the non-commutative C^*-algebra of all (bounded) operators on $H(X, \omega)$ should induce a deformation (in the parameter h) of the commutative C^*-algebra $C^\infty(X, \mathbb{C})$ (this is the subject of *deformation quantization*). See for example [AE05, Gut00] for a general survey on quantization.

As shown by Berezin, Cahen, Gutt, Rawnsley and others any positively curved metric ϕ on a line bundle $L \to X$ induces a quantization with $h = 1/k$, where k is a positive integer. If $\omega = \omega_\phi$ the quantization (at level k) of (X, ω_ϕ) is obtained by letting $H(X, \omega) := H^0(X, L^{\otimes k})$ equipped with the Hermitian metric $Hilb(k\phi)$:

$$Hilb_k(k\phi)(s, \bar{s}) = \int_X |s|^2_{h_0} e^{-k\phi} \frac{\omega_\phi^n}{n!}.$$

To any complex-valued function f one associates the *Toeplitz operator* $T_f^{(k)}$ *on* $H^0(X, L^{\otimes k})$ *with symbol* f. It is defined by the corresponding quadratic form on the Hilbert space $H^0(X, L^{\otimes k})$:

$$\left\langle T_f^{(k)} s, s' \right\rangle_{k\phi} := \left\langle fs, s' \right\rangle_{k\phi}, \tag{2}$$

In other words,

$$T_f^{(k)} = P_k f \cdot, \ \ P_k : \mathcal{C}^\infty(X, L^{\otimes k}) \to H^0(X, L^{\otimes k}),$$

where P_k is the orthogonal projection induced by the Hilbert space structure. In particular, $T_{\bar{f}}^{(k)} = (T_f^{(k)})^*$ and hence $T_f^{(k)}$ is Hermitian if f is real. It turns out that there is an asymptotic expansion (in operator norm) [Eng02, Sch00, MM07]

$$T_f^{(k)} T_g^{(k)} - T_{fg}^{(k)} = \frac{1}{k} T_{c_1(f,g)}^{(k)} + \frac{1}{k^2} T_{c_2(f,g)}^{(k)} + ... \tag{3}$$

where $c_i(f, g)$ is a bi-differential operator (where we have written out $c_0(f, g) = fg$. The corresponding formal induced star product on symbols: $f * g := fg + c_1(f, g)h + ...$ is usually called the *Berezin-Toeplitz star product.* In particular,

$$[T_f^{(k)}, T_g^{(k)}] = \frac{1}{k} T_{c_1(f,g) - c_1(g,f)} + O(1/k^2),$$

where $c_1(f, g) - c_1(g, f)$ is the Poisson bracket on $C^\infty(X, \mathbb{C})$ induced by the symplectic form ω (compare with formula (25)). Moreover, if f is real-valued and $\sigma(T_f^{(k)})$ denotes the spectrum of the operator $T_f^{(k)}$, then

$$\frac{1}{k^n} \sum_{\lambda_i^{(k)} \in \sigma(T_f^{(k)})} \delta_{\lambda_i^{(k)}} \to f_* (\omega_\phi)^n / n! \tag{4}$$

In particular, setting $f = 1$ and integrating over $\mathbb{R}$ gives the asymptotic Riemann-Roch formula

$$N_k := \dim H^0(X, L^{\otimes k}) = k^n \int_X \frac{\omega_\phi^n}{n!} + O(k^{n-1}) \tag{5}$$

which is consistent with the "correspondence principle", since it identifies the leading asymptotics of the *dimension* of the *quantum* state space with the *volume* of the *classical* phase space. All of these results may be deduced from the asymptotic properties of the *Bergman kernel* $K_k(x, y)$, i.e. the integral kernel of the orthogonal projection P_k. These asymptotics were obtained by Catlin and Zelditch [Cat99, Zel98] using the micro-local analysis of Boutet de Monvel-Sjöstrand. There are by now several approaches to these asymptotics; see the review article [Zel09] for an introduction and references. In particular, there is an asymptotic expansion of the point-wise norm $\rho(k\phi)(x)$ of $K_k(x, x)$ (also called the "distortion function" or "density of states function"):

$$k^{-n} \rho(k\phi)(x) = (2\pi)^{-n} (1 + k^{-1} b_1(x) + k^{-2} b_2(x) + ...) \tag{6}$$

which holds in the $\mathcal{C}^\infty-$topology and where the coefficients b_i depend polynomially on ϕ and its derivatives. Finally, it should be pointed out that twisting $L^{\otimes k}$ with an Hermitian holomorphic line bundle L' only has the effect of changing the expression for the coefficients in the expansions above.

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1 The results of D.H. Phong and J. Sturm

1.1 The homogeneous Monge-Ampère equation and geodesics in $\mathcal{H}_\infty$

Given a smooth metric ϕ on $L \to X$ its Monge-Ampère is the form $(\omega_\phi)^n/n!$ of maximal degree on X. In particular, $MA(\phi) \geq 0$ if ϕ has semi-positive curvature, i.e. if $\omega_\phi \geq 0$. As shown in the seminal work of Bedford-Taylor $MA(\phi)$ is naturally defined as (positive) measure for any ϕ which is locally bounded with $\omega_\phi \geq 0$ in the sense of currents.

Given $\phi_0, \phi_1 \in \mathcal{H}_\infty$ the geodesic ϕ_t from the introduction may be obtained as follows from a complex point of view. Firstly, let us complexify the variable t to take values in the strip $[0,1]+\sqrt{-1}[0,2\pi]$ which we will identify, as a complex manifold, with the closure of an annulus A in $\mathbb{C}$. Then pull-back ϕ_0 and ϕ_1 to S^1-invariant functions on the boundary of $M := X \times A$. Pulling back ω_0 from X induces a semi-positive form $\pi^*\omega_0$ on M. Now denote by $\Phi = \phi_t(\cdot)$ the function on $\overline{M}$ obtained as the unique solution of the following Dirichlet problem: $\Phi \in C^0(\overline{M})$, where $\Phi_{\partial M}$ coincides with the given data above and

$$MA_{(x,t)}\Phi := (\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi(x,t))^{n+1} = 0, \quad (x,t) \in M, \tag{7}$$

with $\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi(x,t) \geq 0$. As shown by Chen [Che00] (see also [Blo09]) the solution Φ is unique and almost C^2-smooth, in the sense that $\partial\bar{\partial}\Phi(x,t)$ has locally bounded coefficients. Note that expanding gives the relation

$$MA_{x,t}(\Phi) = c(\phi_t)(dt \wedge d\bar{t} \wedge MA_x(\phi_t)), \tag{8}$$

where $c(\phi_t)$ is the geodesic curvature in $\mathcal{H}_\infty$ (compare with formula (1)). Hence, if (i) ϕ_t is in $C^\infty(X)$ and (ii) $\partial_t\bar{\partial}_t\phi_t > 0$ for all t, then equation (7) is equivalent to

$$c(\phi_t) := \ddot{\phi}_t - |\bar{\partial}_x\dot{\phi}_t|^2_{\omega_\phi} = 0 \tag{9}$$

i.e. ϕ_t is a geodesic in n $\mathcal{H}_\infty$. However, it should be pointed out that it is still not known whether any of the two conditions above hold in general. Hence, the "geodesic" ϕ_t obtained above is a path in the *closure* of $\mathcal{H}_\infty$.

Remark 1. The situation becomes considerably simpler in the *toric* setting, i.e. when the real $n-$torus T^n acts with an open dense orbit on X and equivariantly on $L \to X$ and $\mathcal{H}_\infty$ is replaced by the space $\mathcal{H}_\infty^{T^n}$ of all T^n-invariant metrics. In this setting $\phi(x)$ may be represented by a convex function on $\mathbb{R}^n_x$. For any t we may consider the Legendre transform $\psi_t(p) = \phi_t^*(p)$ of $\phi_t(x)$ which is a convex function on the dual real vector space $(\mathbb{R}^n)^*$. Then equation (9) is equivalent to the equation

$$\ddot{\psi}_t(p) = 0,$$

i.e. ψ_t is simply the *affine* interpolation of ψ_0 and ψ_1 [Gua99, Theorem 3]. An important conceptual feature of the fiber-wise Legendre transformation is that the transform function ψ_t satisfies a differential equation depending only on the $t-$variable.

1.2 A canonical functional

The measure valued operator $\phi \mapsto MA(\phi)$ on the space $\mathcal{H}_\infty$ may, in the standard way, be identified with a differential one-form MA on $\mathcal{H}_\infty$, using that $\mathcal{H}_\infty$ is a convex subset of the affine space $C^\infty(X)$. As observed by Mabuchi this one-form is in fact *exact* [Mab87]. Equivalently, there exists a functional $\mathcal{E} : \mathcal{H}_\infty \to \mathbb{R}$ (the "primitive" of MA) such that $d\mathcal{E}_{|\phi} = MA(\phi)$ or equivalently

$$\frac{d}{dt}\mathcal{E}(\phi_t) = \int_X \frac{d}{dt}\phi_t MA_x(\phi_t). \tag{10}$$

for any curve ϕ_t in $\mathcal{H}_\infty$ (see [Aub84, Section III] and [Yau78, Section 2]). If one imposes the normalization $\mathcal{E}(0) = 0$ the functional $\mathcal{E}$ is hence uniquely determined. Working now on $X \times A$, a direct computation shows that

$$\partial_t \bar{\partial}_t \mathcal{E}(\Phi(x,t)) = \int_{t \in A} MA_{x,t}(\Phi), \tag{11}$$

just using Leibniz rule and the relation (8). Hence, it follows directly from the homogeneous Monge-Ampère equation (7) that the following proposition holds:

Proposition 2. *The following properties of the functional $\mathcal{E}$ hold:*

- *If ϕ_t is a geodesic in $\mathcal{H}_\infty$, then $\mathcal{E}(\Phi(.,t))$ is affine with respect to t real.*
- *If the metric $\Phi(z,t)$ on $\pi^* L \to X \times A$ has semi-positive curvature, then $t \mapsto \mathcal{E}(\Phi(.,t))$ is subharmonic with respect to t. Hence, $\mathcal{E}(\Phi(.,t))$ is convex along geodesics.*

1.3 Quantification scheme

Let $H^0(kL) = H^0(X, L^{\otimes k})$ be the space of all global holomorphic sections of $kL := L^{\otimes k}$ over X. We will denote by N_k the dimension of this complex vector space of finite dimension (since M is compact). Let $\mathcal{H}_k$ be the set of all Hermitian metrics H on the vector space $H^0(kL)$, the *Bergman space at level* k. The map $A \mapsto A^*A$ clearly yields an isomorphism

$$GL(N_k, \mathbb{C})/U(N_k) \simeq \mathcal{H}_k \tag{12}$$

turning $\mathcal{H}_k$ into a symmetric space. We can define two natural maps :

- ("quantization"). The 'Hilbert' map

  $$Hilb_k : \mathcal{H}_\infty \to \mathcal{H}_k$$

  such that

  $$Hilb_k(k\phi)(s, \bar{s}) = \int_X |s|^2 e^{-k\phi} MA(\phi).$$

- ("dequantization"). The injective map 'Fubini-Study',

  $$FS_k : \mathcal{H}_k \to \mathcal{H}_\infty$$

  such that

  $$FS_k(H) = \frac{1}{k} \log(\sum_{i=1}^{N_k} |s_i^H|^2_{h_0}),$$

  where (s_i^H) is an H-orthonormal basis[1] of holomorphic sections of $H^0(kL)$.

We will often identify $\mathcal{H}_k$ with its image in $\mathcal{H}_\infty$ under FS_k and call it the space of *Bergman type metrics* of order k. Geometrically, $FS_k(H)$ is just the scaled pull-back of the Fubini-Study metric on $\mathcal{O}(1) \to \mathbb{P}H^0(kL)$, induced by H under the Kodaira embedding $x \mapsto [s_1^H(x) : ... : s_{N_k}^H(x)]$. More invariantly, this is the natural "evaluation map" $x \mapsto \mathbb{P}H^0(kL)^\vee$ composed with the isomorphism between $\mathbb{P}H^0(kL)^*$ and $\mathbb{P}H^0(kL)$ determined by H.

Next, we recall the following fundamental approximation result first proved by Bouche [Bou90] and Tian [Tia90] (see [Rua98] for the issue of smooth convergence).

Proposition 3. *Let ϕ be an element in $\mathcal{H}_\infty$, i.e. a smooth metric on L with positive curvature. When $k \to \infty$ the composed maps $P_k := FS_k \circ Hilb_k$ approximate the identity. More precisely,*

$$FS_k \circ Hilb_k(k\phi) \to \phi \tag{13}$$

in the $\mathcal{C}^\infty-$topology.

[1] note that this definition is independent of the choice of the basis.

In fact, if ϕ is only assumed continuous then a simply approximation arguments gives $\mathcal{C}^0-$convergence. The previous proposition is a direct consequence of the leading asymptotics of $\rho(k\phi)(x)$ in (6) since

$$\rho(k\phi)(x) := \sum_{i=1}^{N_k} |s_i^{Hilb_k(k\phi)}(x)|^2_{h_0} e^{-k\phi(x)} = e^{kFS_k(Hilb_k(k\phi))(x)} e^{-k\phi(x)}$$

so that taking log, dividing by k and letting $k \to \infty$ proves the proposition.

1.4 Geodesics in the Bergman spaces

Let us fix k and let $H_0, H_1 \in \mathcal{H}_k$ be two Bergman metrics. By standard linear algebra there exist numbers λ_i with $1 \leq i \leq N_k$ and bases $(s_i^{H_0})$ and $(s_i^{H_1})$, orthonormal with respect to H_0 and H_1 respectively, such that

$$s_i^{H_1} = s_i^{H_0} e^{\lambda_i/2}$$

The geodesic H_t in $\mathcal{H}_k$ (with respect to the Riemann structure induced by the isomorphism (12)) between H_0 and H_1 may then be concretely obtained in the following way: H_t is the Hermitian metric such that

$$s_i^{H_t} = s_i^{H_0} e^{t\lambda_i/2}$$

is H_t-orthonormal. We are now ready to state Phong and Sturm's result [PS06, PS05].

Theorem 4 (Phong-Sturm, 2005)**.** *Let ϕ_t be the unique $C^{1,1}$ geodesic from ϕ_0 to ϕ_1 in $\overline{\mathcal{H}_\infty}$. Let $H_t^{(k)}$ be a Bergman geodesic curve in $\mathcal{H}_k$ such that $H_0^{(k)} = Hillb_k(k\phi_0)$ and $H_1^{(k)} = Hilb_k(k\phi_1)$. Then $\Phi^{(k)} := FS_k(H_t^{(k)})$, identified with a metric over $X \times A$, satisfies*

$$(\sup_{k \geq l} \Phi^{(k)})^{*usc} \to \Phi$$

uniformly over $X \times A$ as $l \to \infty$.

Here, one has defined the upper-envelope of a bounded function $u : X \times [0,1] \to \mathbb{R}$, by setting

$$u^{*usc}(z) = \lim_{\epsilon \to 0} \sup_{|z'-z|<\epsilon} u(z')$$

Recall that a sequence of plurisubharmonic functions u_k which are locally uniformly bounded, $(\sup u_k)^{*usc}$ is still plurisubharmonic and equal to $\sup u_k$ almost everywhere. The proof of Phong-Sturm uses the result established by Chen [Che00] concerning the existence and regularity of the geodesic Φ in the (closure of) the space of Kähler potentials $\mathcal{H}_\infty$. More precisely, it is the $\mathcal{C}^0-$regularity of Φ which is needed (this fact immediately gives uniform convergence in the theorem above, by Dini's lemma). As recently observed in [BD09] this latter regularity can also be obtained by adapting the approach of Bedford-Taylor for pseudoconvex domains in $\mathbb{C}^n$ to the present situation.

1.4.1 The proof of Theorem 4

We keep the notation $\Phi^{(k)} := FS_k(H^{(k)})$ for the metric over $X \times A$ induced by rotational symmetry from $FS_k(H_t^{(k)})$ on X. The two main ingredients in the proof of Phong and Sturm are as follows. Firstly the following uniform estimate in k on $X \times A$:

$$\left| \frac{\partial}{\partial t} \Phi^{(k)} \right| \leq C \tag{14}$$

Secondly, the "volume estimate"

$$MA_{x,t}(\Phi^{(k)}) \to 0. \tag{15}$$

weekly on the interiour of $X \times A$. The estimate (14) is used to control the boundary behaviour of $\Phi^{(k)}$ (by Proposition 3, convergence towards Φ at the boundary is already clear). Moreover, from the convergence (15), any limit point of $\Phi^{(k)}$ satisfies the homogeneous Monge-Ampère equation in the interior of $X \times A$. By adapting the pluripotential results of Bedford-Taylor for pseudoconvex domains in $\mathbb{C}^n$ to the their situation, Phong-Sturm finally conclude the proof of Theorem 4. Finally, the last difficulty is to establish a suitable uniqueness result for "rough" solutions of the Dirichlet problem (7).

The proof of (14) uses the explicit formula

$$\frac{\partial}{\partial t} \Phi^{(k)} = \sum_{j=1}^{N_k} \lambda_j |s_j^{H_t^k}|^2 e^{-FS_k(H_t^k)}$$

to reduce the estimate to a uniform bound on $\frac{2}{k} \max |\lambda_j|$ (see [PS06, Lemma 1]). Note that the *upper* bound in (14) (without the absolute values) is a direct consequence of the convexity of the map $t \mapsto \Phi^{(k)}$ for t real, combined with the uniform bounds at the end points $t = 0, 1$ furnished by Proposition 3.

Now, the estimate (15) can be proved by first noting that, by the second point in Proposition 2, it is a consequence of the fact that

$$\int_{X \times A} MA_{x,t}(\Phi^{(k)}) \to 0 \tag{16}$$

when $k \to \infty$. But one has by (10) and (11),

$$\begin{aligned} \int_{X \times A} MA_{x,t}(\Phi^{(k)}) = \quad & \int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=1} MA((\Phi_{t=1}^{(k)})) \\ & - \int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=0} MA(\Phi_{t=0}^{(k)}). \end{aligned} \tag{17}$$

If we let $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$, we can write

$$\int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=1} MA(FS_k(\Phi^{(k)}_{t=1})) = \frac{1}{k}\int_X \sum_{j=1}^{N_k} \lambda_j |s_j^{H_1^k}|^2 e^{-FS_k(H_1^k)} MA(FS_k(H_1^{(k)}))$$

$$= \frac{1}{k}\int_X \sum_{j=1}^{N_k} \lambda_j |s_j^{H_1^k}|^2 e^{-k\phi_1} \frac{e^{-kFS_k(H_1^k)}}{e^{-\phi_1}} \frac{MA(FS_k(H_1^{(k)}))}{\omega_{\phi_1}^n} \omega_{\phi_1}^n.$$

But for $i = 0, 1$, Proposition 3 gives

$$\frac{MA_x(FS_k(H_i^{(k)}))}{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_i)^n} = 1 + O(1/k), \qquad \frac{e^{-FS_k(H_i^k)}}{e^{-\phi_i}} = 1 + O(1/k).$$

Combining this with the bound (14) gives

$$\begin{aligned}
\int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=1} MA(FS_k(\Phi^{(k)}_{t=1})) &= \frac{1}{k}\sum_{j=1}^{N_k} \int_X \lambda_j |s_j^{Hilb_k(e^{-k\phi_1})}|^2 e^{-k\phi_1} \omega_{\phi_1}^n \\
&\quad + O\left(\frac{1}{k}\right) \\
&= \frac{1}{k}\sum_{j=1} \lambda_j + O\left(\frac{1}{k}\right).
\end{aligned}$$

Repeating the argument for $t = 0$ also gives

$$\int_X \frac{\partial \Phi^{(k)}}{\partial t}_{|t=0} MA(FS_k(\Phi^{(k)}_{t=1})) = \frac{1}{k}\sum_{j=1} \lambda_j + O\left(\frac{1}{k}\right).$$

All in all this proves (16) and hence finishes the proof of (15).

2 The results of B. Berndtsson

In [Ber09b], B. Berndtsson develops a different approach that we discuss now. He considers not the spaces $H^0(kL)$ but instead the spaces $H^0(kL + K_X)$ of all holomorphic $n-$forms with values in kL. We now redefine $N_k = \dim H^0(X, kL + K_X)$ and

$$\mathcal{H}_k = \{\text{smooth hermitian metrics on } H^0(kL + K_X)\},$$

and also

$$Hilb_k : \mathcal{H}_\infty \to \mathcal{H}_k$$

by

$$Hilb_k(k\phi)(s, \bar{s}) = \int_X |s|_h^2 e^{-k\phi} dz \wedge d\bar{z}.$$

A technical difficulty is to redefine the Fubini-Study map $FS_k : \mathcal{H}_k \to \mathcal{H}_\infty$ in this setting. We are lead to tensorize $\frac{1}{k}\log(\sum |S_i^H|^2)$, seen as a metric over $L + \frac{1}{k}K_X$, by a metric from $-\frac{1}{k}K_X$. This later metric is fixed *a priori*, and we will hence be able to control it uniformly when $k \to +\infty$. Finally, we continue to denote $H^{(k)}(t) = H_t^{(k)}$ the Bergman geodesic in $\mathcal{H}_k$ between $H_0^{(k)}$ and $H_1^{(k)}$ defined as in the previous section.
As it turns out the introduction of the canonical line bundle in the Bergman space will considerably simplify the estimates. The reason is that the corresponding L^2-estimates for the $\bar{\partial}-$equation of Hörmander and Kodaira are sharp in this setting.

Theorem 5 (Berndtsson, 2006)**.** *, Given two metrics $\phi_0, \phi_1 \in \mathcal{H}_\infty$, there exists $\Phi = \phi_t \in C^0(X \times A)$ such that $FS_k(H^{(k)}) \to \Phi$ in the $C^0(X \times A)$ topology. More precisely,*

$$\sup_{X\times A} |FS_k(H^{(k)}) - \Phi| \leq C \log k/k$$

Moreover,

$$\Phi = \sup_\Psi \{\Psi : \Psi \leq \Phi \text{ on } \partial(X \times A)\}, \tag{18}$$

where the sup is taken over all strictly positively curved smooth metrics Ψ on the pulled back line bundle $\pi^*L \to X \times A$.

As very recently shown by Berndtsson in [Ber09c] a simple modification of the proof of Theorem 5 shows that it is more generally valid for $kL + L'$ where L' is any Hermitian holomorphic line bundle equipped with a smooth metric (possibly depending on ϕ). In particular, it applies to the setting of Phong-Sturm. For simplicity we will mainly stick to the case of $kL + K_X$.

Remark 6. The relation to continuous geodesics or more precisely continuous solutions of the Dirichlet problem (7) is not explicitly discussed in [Ber09b], but is essentially well-known for any manifold M with boundary (here $M := X \times A$). Indeed, by using a family version of Richberg's classical approximation result, the sup defining Φ may be taken over all Ψ which are merely continuous (and with semi-positive curvature current). Then, by solving *local* Dirichlet problems on any small ball in the interior of M, one sees that $MA(\Phi) = 0$ in the interior of M, by following Bedford-Taylor. Conversely, any *continuous* solution of the global Dirichlet problem on M is necessarily *maximal* in M by the maximum principle for the Monge-Ampère operator:

$$\Phi \geq \Psi \text{ on } \partial M \ \Rightarrow \Phi \geq \Psi \text{ on } M$$

if Ψ is continuous with semi-positive curvature (this part is elementary and proved exactly as in the $\mathbb{C}^n-$setting, see for example Lemma 3.7.2 in [Kli91]). In particular, the solution is *unique* and of the form (18). It should be pointed out that, as opposed to Phong-Sturm's proof, the proof of Berndtsson does not rely on any regularity results concerning the solution of the Dirichlet problem (7). In fact, it gives a new interesting "constructive" proof of the C^0-regularity in this setting.

The key point in the proof of Berndtsson's theorem is the following "quantized maximum principle" that we shall explain in Section 2.1.1.

Proposition 7 (Quantized maximum principle)**.** *If ψ_t is plurisubharmonic on $X \times A$ and if $Hilb_k(k\psi_t) \geq H_t^{(k)}$ for $t \in \partial A$ with $H_t^{(k)}$ geodesic in $\mathcal{H}_k$, then*

$$Hilb_k(k\psi_t) \geq H_t^{(k)}$$

for all $t \in A$.

We explain now how Proposition 7 implies the convergence of the sequence of Bergman geodesics. We set $\phi_t^{(k)} = FS_k(H_t^{(k)})$ which is positively curved over $X \times A$, as follows immediately from its explicit expression. Let us show that the following two inequalities hold:

$$\phi_t^{(k)} \leq \phi_t + O(\log k/k), \tag{19}$$

$$\phi_t \leq \phi_t^{(k)} + O(1/k). \tag{20}$$

For (19), we notice that on $\partial(X \times A)$, Proposition 3 gives

$$\phi_t^{(k)} := FS_k(H_t^{(k)}) = FS_k(Hilb_k(k\phi_i)) \leq \phi_i + O(\log k/k)$$

with $i = 0$ or $i = 1$. Now, from the extremal definition of ϕ_t, we get (19) on all of $X \times A$, using that $FS_k(H_t^{(k)})$ has semi-positive curvature as a metric over $X \times A$. Note that the upper bound $O(\log k/k)$ is actually sharp.

For (20), we will now use Proposition 7. On $\partial(X \times A)$, since $Hilb_k(k\phi(., t)) = H_t^{(k)}$, and because any given candidate Ψ for the sup defining Φ is positively curved over $X \times A$, we obtain on all of $X \times A$,

$$Hilb_k(k\Psi) \geq H_t^{(k)}.$$

which implies

$$FS_k(Hilb_k(k\Psi)) \leq \phi_t^{(k)}.$$

Finally, we obtain (20) if we can prove that for any smooth metric ψ on L with semi-positive curvature form

$$\psi \leq FS_k(k\psi) + \frac{c}{k} \tag{21}$$

with a uniform constant c independent of ψ. This estimate is a well-known consequence of the celebrated Ohsawa-Takegoshi theorem. We briefly recall a weak version of this latter result in the following

Proposition 8 (Ohsawa-Takegoshi). *Let $L \to X$ be an ample line bundle, ψ a smooth metric on L such that $\omega_\psi \geq 0$ and fix $x \in X$. Then, for any $k >> 0$, there exists a holomorphic section $s_k \in H^0(kL + K_X)$ such that $|s_k(x)|^2_{k\psi} = 1$ and $\int_X |s_k|^2_{k\psi}\omega^n \leq C$ with C independent of ψ , x and s_k.*

The estimate (21) comes by using that

$$FS_k(Hilb_k(k\psi))(x) - \psi \geq \frac{1}{k}\log\left(\frac{|s_k(x)|^2_{k\psi}}{\int_X |s_k|^2_{k\psi}\omega^n}\right)$$

for any section s_k and in particular for the one, depending on x, furnished by the proposition.

2.1 Curvature of direct image bundles

We will next turn to the proof of the crucial "quantified" maximum principle of Berndtsson. The starting point of its proof is the following geometric description of a geodesic $H_t^{(k)}$ in $\mathcal{H}_k$. It may be suggestively described in the "quantization" terminology. A curve $\phi_t \in \mathcal{H}_\infty$ gives rise to a familly (X, ω_{ϕ_t}) of Kähler manifolds fibred over $[0,1]_t$. Its quantization is hence a family of Hermitian vector spaces $(H^0(kL + K_X), Hilb_k(k\phi_t))$ fibred over $[0,1]_t$. This is equivalent to say that, by complexifying the parameter t so that it lives in the disk annulus A of $\mathbb{C}$, we arrive at the following suggestive statement: *the quantization of a curve ϕ_t in $\mathcal{H}_\infty$ is a holomorphic Hermitian vector bundle $(E, \mathbf{H})$ over A, which is holomorphically isomorphic to $H^0(kL + K_X) \times A$.*

Similarly, any curve $H_t \in \mathcal{H}_k$ gives rise to a vector bundle $(E, \mathbf{H})$ over A. As observed by Berndtsson H_t is a geodesic in $\mathcal{H}_k$ precisely when the curvature $\Theta_E(\mathbf{H})$ of the vector bundle vanishes,

$$\Theta_E(\mathbf{H}) = 0 \in End(E).$$

Recall the following general definition of curvature [GH94]: If H_t is a family of Hermitian matrices locally representing an Hermitian metric on a holomorphic vector bundle $E \to A$, then the (Chern) connection form $\theta(H_t)$ is the following local matrix valued $(0,1)-$form on A :

$$\theta(H_t) = -H_t^{-1}\partial_t H$$

Its curvature (which in our convention is "real") is the following local matrix valued real $(1,1)-$form on A :

$$\Theta_E(\mathbf{H}) = \Theta_E(H_t) = \sqrt{-1}\bar{\partial}_t\theta(H_t) = -\sqrt{-1}\bar{\partial}_t(H_t^{-1}\partial_t H)$$

defining a global $(1,1)$-form on A with values in $End(E)$. In our case the base A is one-dimensional and E is holomorphically trivial with fiber $H^0(kL + K_X)$. Hence we may and will identify $\Theta_E(H_t)$ with an Hermitian operator on $H^0(kL + K_X)$ for any $t \in A$.

We next state a fundamental result of B. Berndtsson [Ber09b] about the curvature of the vector bundle obtained by quantizing a curve ϕ_t in $\mathcal{H}_\infty$.

Theorem 9. *If Ψ is a metric on $\pi^*L \to X \times A$ such that $\psi_t := \Psi_t(\cdot) \in \mathcal{H}_\infty$, then*

$$\langle \sqrt{-1}\Theta_E \left(\mathbf{Hilb_k}(k\psi_t)\right) s, s\rangle \quad \geq \quad k\langle c(\psi_t)s, s\rangle$$

in terms of the geodesic curvature $c(\psi_t)$ of ψ_t (compare with formula (8)) and where the inner product is with respect to $\mathbf{Hilb_k}(k\psi_t)$. In particular, if Ψ is positively curved over $X \times A$ then

$$\sqrt{-1}\Theta_E(\mathbf{Hilb_k}(k\psi_t)) \geq 0$$

as an Hermitian form.

2.1.1 The proof of Theorem 9

The proof of Berndtsson theorem[2] takes advantage of the fact that the hermitian holomorphic vector bundle E over A is a subbundle of the (infinite dimensional) hermitian holomorphic vector bundle F, where $F_t := \mathcal{C}^\infty(X, kL + K_X)$ whose fiber at t consist of all *smooth* sections. If one endows F with the holomorphic structure defined by the operator $\bar{\partial}_t$ (the $\bar{\partial}-$operator along the base) then E clearly becomes an Hermitian *holomorphic* subbundle of F. A simple calculation gives the connection "matrix" (i.e. a linear operator)

$$(\theta_F)_t := -\frac{\partial\Psi}{\partial t} = -u_t\cdot, \qquad u_t := \dot{\psi}_t$$

Moreover, well-known formulas for induced connections on holomorphic subbundles (see p. 78 in [GH94]) give

$$(\theta_F)_t = P_t(\theta_E) = -T^{(k)}_{u_t},$$

where P_t is the orthogonal projection $F_t \to E_t$, using the notation of Toeplitz operators in the introduction. Moreover, the curvature on the subbundle E may be expressed as (see [GH94])

$$\Theta_E = P\Theta_F - B(u)^*B(u) =_{|t} P_t(\dot{u}_t\cdot) - B(u_t)^*B(u_t) \tag{22}$$

where $B(u)$ is the linear operator ("second fundamental form") $B(u_t)s = u_ts - P_t(u_ts)$. We next formulate the key estimate of Berndtsson in the following result.

Lemma 10. *The following inequality holds for any smooth function u and smooth metric ψ on L with positive curvature:*

$$\langle B(u)^*B(u)s, s\rangle_{k\psi} \geq \left\langle |\bar{\partial}_x u|^2_{\omega_\psi} s, s\right\rangle_{k\psi}$$

for all $s \in H^0(X, kL + K_X)$.

[2] and in particular the proof of the more general curvature estimate in [Ber09b].

Proof. First note that

$$\langle B^*Bs, s\rangle_t = \langle Bs, Bs\rangle_t := \|us - P(us)\|_t^2$$

But since P is the orthogonal projection onto the kernel of $\bar{\partial}_x$, the smooth section $v := us - P(us)$ of $kL + K_X$ is the element with minimal norm of the following inhomogeneous $\bar{\partial}$-equation

$$\bar{\partial}_x v = \bar{\partial}_x(us).$$

Hence, by the L^2-estimates of Hörmander-Kodaira

$$\|v\|_{k\psi}^2 \leq \left\|\bar{\partial}_x(su)\right\|_{k\psi,\omega_\psi}^2$$

where $\bar{\partial}_x(su)$ is an $(n, 1)-$form with values in kL and where ω_ψ is used to measure the $(0, 1)-$part in the usual way. □

To conclude the proof of Theorem 9 recall that $u_t = \dot{\psi}_t$. Hence formula (22) combined with the previous lemma proves the first statement of Theorem since $c(\psi_t) = \ddot{\psi}_t - |\bar{\partial}_x\dot{\psi}_t|^2_{\omega_\psi}$.

The proof of Proposition 7, is now a direct consequence of Theorem 9 and the following lemma (compare with [CS93]).

Lemma 11. *Let E be a holomorphic vector bundle on a smooth domain $\overline{D}$ in $\mathbb{C}$ and let H_0, H_1 be two hermitian metrics on E that extend continuously to $\overline{D}$. Assume that the curvature of H_0 is flat and the curvature of H_1 is semi-positive. If, $H_0 \leq H_1$ on ∂D, then $H_0 \leq H_1$ in D.*

Proof. For the sake of completeness we give a simple proof of the lemma, which was explained to us by Bo Berndtsson. As is well-known E is holomorphically trivial on D (anyway we will only apply the lemma to a trivial bundle). We fix a global trivialization of E. Let s be a given global holomorphic section of E over $\overline{D}$. By the usual maximum principle for the Laplace operator it will be enough to prove the following claim: $|s|^2_{H_0} - |s|^2_{H_1}$ is a subharmonic function on D. To this end we will use the following basic formula, which follows from an application of Leibniz rule,

$$\frac{\partial^2}{\partial t\partial\bar{t}}(|s|^2_H) = -\sqrt{-1}\,\langle\Theta_H s, s\rangle_H + \left\langle D_H^{1,0}s, D_H^{1,0}s\right\rangle_H,$$

where $D_H^{1,0} = \frac{\partial}{\partial t} - \theta(H)$, is the $(1, 0)-$component of the Chern connection determined by H. Now, as it is well-known, for any given fixed point $t \in D$ we can always assume that $D_{H_1}^{1,0}s = 0$ at t, after perhaps performing a gauge transformation, i.e. after replacing s with gs and H_1 by gH_1g^{-1}, where g is a holomorphic function on $\overline{D}$ (depending on the fixed point t) with values in the space of invertible matrices. Since, $|s|^2_H$ and the curvature Θ_H are invariant under gauge transformations the previous formula gives, at the fixed point t,

$$\begin{aligned}\frac{\partial^2}{\partial t \partial \bar{t}}(|s|^2_{H_0} - |s|^2_{H_1}) &= -\sqrt{-1}\left\langle \Theta_{H_0} s, s\right\rangle_{H_0} \\ &\quad + \left\langle D^{1,0}_{H_0} s, D^{1,0}_{H_0} s\right\rangle_{H_0} + \sqrt{-1}\left\langle \Theta_{H_1} s, s\right\rangle_{H_1} + 0 \\ &\geq 0,\end{aligned}$$

using the curvature assumptions. Since the point t was arbitrary this finally proves the claim above and hence finishes the proof of the lemma. □

2.2 Concluding remarks

2.2.1 The correspondence principle and curvature asymptotics

In the spirit of the "correspondence principle" referred to in the introduction, one may rewrite the lower bound in Theorem 9 as the inequality

$$\sqrt{-1}\Theta_E\left(\mathbf{Hilb_k}(k\psi_t)\right) \geq T^{(k)}_{c(\psi_t)} \tag{23}$$

between two Hermitian operators on $H^0(X, kL + K_X)$, i.e. *"the quantization of the geodesic curvature of a curve ϕ_t is always smaller than the curvature of the quantization of ϕ_t"*! (see the discussion in [Ber09a]).

In fact, using essentially well-known asymptotic formulas one can show that one obtains an *equality* in (23), up to lower order terms in "Planck's constant" $h = 1/k$. To see this first note that a simple computation gives, with notation as in Lemma 10, the following expression in terms of Toeplitz operators:

$$-B(u)^*B(u) = (T^{(k)}_u)^2 - T^{(k)}_{u^2}$$

Now we can use the asymptotic expansion from the introduction:

$$T^{(k)}_f T^{(k)}_g - T^{(k)}_{fg} = \frac{1}{k}T^{(k)}_{c_1(f,g)} + O(\frac{1}{k^2}) \tag{24}$$

in operator norm, with the explicit formula

$$c_1(f,g) = \sqrt{-1}(\partial f \wedge \bar{\partial} g \wedge \omega^{n-1}_\phi)/\omega^n_\phi. \tag{25}$$

(this follows for example from explicit formula for c^{BT}_1 in [Eng02]). Setting $f = g = u$ shows that (23) is an asymptotic *equality*, i.e.

$$k^{-1}\sqrt{-1}\Theta\left(\mathbf{Hilb_k}(k\psi_t)\right) = T_{c(\psi_t)} + O(1/k) \tag{26}$$

in operator norm. In particular, $\sqrt{-1}\Theta(\mathbf{Hilb_k}(k\psi_t)) > 0$ for $k > k_0$ if Ψ is smooth with strictly positive curvature (where k_0 depends on Ψ). Hence, the previous asymptotics could be used as a substitute for the curvature estimate in Theorem 9 and in the proof[3] of Theorem 5. Since the asymptotics (24) are still valid with the same formula for $c_1(f,g)$ when kL is twisted by a Hermitian holomorphic line bundle L' this argument also gives uniform convergence in the setting of Phong-Sturm.

Note also that, combined with the asymptotics (4) for spectral measures, (26) implies the spectral asymptotics

$$\frac{1}{k^n}\sum_i \delta_{\lambda_i^{(k)}} \to c(\psi_t)_*(\omega_\phi)^n/n!$$

converging in the weak star topology of measures, summing over all eigenvalues of the curvature $\sqrt{-1}\Theta\left(\mathbf{Hilb_k}(k\psi_t)\right)$. It should also be pointed out that asymptotics for curvature operators on very general direct image bundles have been announced by Ma-Zhang in [MZ07] (without explicitly using the relation to Toeplitz operator asymptotics).

2.2.2 Wess-Zumino-Witten type equations

It may be of some interest to point of that the proof of Theorem 5 extends almost word for word to the case when the geodesic is replaced by a solution to certain Wess-Zumino-Witten type equations (see [Don99] for the relation between geodesics and Wess-Zumino-Witten type equations). To setup the problem first consider the following generalization of the Dirichlet problem (7). Let us replace the annulus A with a general smooth domain D in $\mathbb{C}$ and let us assume given a continuous function $\Phi = \phi_t(x)$ on ∂D such that $\phi_t \in H_\infty$ for all $t \in \partial D$. Then there is a unique continuous extension of Φ to D such that

$$MA_{x,t}(\Phi) = 0, \ \ (t,x) \in X \times D \tag{27}$$

with $\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi \geq 0$ on $X \times D$ [Che00, BD09]. There is also a "quantized" version of this problem where one assumes given a continuous family $\mathbf{H} = H_t \in \mathcal{H}_k$ for $t \in \partial D$. Then there exists a unique continuous extension of $\mathbf{H}$ from ∂D to a *flat* metric on $E \to D$, i.e. such that

$$\Theta_E(\mathbf{H})_t = 0 \in \mathrm{End}(E) \ \ \forall t \in D$$

[CS93, Don92]. When D is the unit-disc this amounts to classical results of Birkhoff, Grothendieck, Wiener and Masni. Note that this latter equation is a Laplace type PDE in t for a matrix H_t, which is quadratic in the first derivatives of H_t. Now the proof of Theorem 5 may be repeated essentially word for word, showing that the FS_k-images of the flat extensions of $H_t^{(k)} := Hilb_k(k\phi_t)$ converge in the large k limit uniformly on $X \times D$ to a solution Φ of the Dirichlet problem (27) on $X \times D$. The only new ingredient needed is the observation that $\Phi^{(k)} := FS_k(H_t^{(k)})$ still satisfy the condition $\pi^*\omega_0 +$

[3] at least to get uniform convergence, but without the explicit rate of convergence.

$\sqrt{-1}\partial\bar{\partial}\Phi^{(k)} \geq 0$ on $X \times D$. To see this, first recall the following general fact. The curvature $\Theta_E(\mathbf{H})_t$ is semi-positive over D precisely when $\log \|s^*\|^2_{H_t^*}$ is plurisubharmonic in (t, s^*) on the total space of the dual bundle $E^* - \{0\}$, where H_t^* denotes the fiber-wise dual Hermitian metric[4]. Moreover, by definition

$$FS_k(H_t)(x) := \log \|\Lambda_x\|^2_{H_t^* \otimes h_0^*},$$

where Λ_x is the holomorphic section of $E^* \otimes L_x^*$ naturally induced by the pointwise evaluation functional $ev_x : H^0(L) \to L_x$. In particular, $FS_k(H_t)(x)$ is $\pi^*\omega_0-$psh on $D \times X$ proving the observation above.

Finally, it seems natural to ask how to approximate solutions to the Dirichlet problem (27) when D is a domain in $\mathbb{C}^m$ for $m > 1$? As is it well-known one has also to impose the condition that D is *pseudoconvex* in order to get a continuous solution [BD09]. The model case is when D is the unit ball. This leads one to look for a Monge-Ampère type equation for a metric $\mathbf{H}$ on a holomorphic vector bundle $E \to D$ over an $m-$dimensional base. As it turns out, some of the results above do generalize to this higher dimensional setting. Details will hopefully appear elsewhere in the future.

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